

SEMISTABLE VECTOR BUNDLES AND TANNAKA DUALITY FROM A COMPUTATIONAL POINT OF VIEW

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ABSTRACT. We develop a semistability algorithm for vector bundles which are given as a kernel of a surjective morphism between splitting bundles on the projective space \mathbb{P}^N over an algebraically closed field K . This class of bundles is a generalization of syzygy bundles. We show how to implement this algorithm in a computer algebra system. Further we give applications, mainly concerning the computation of Tannaka dual groups of stable vector bundles of degree 0 on \mathbb{P}^N and on certain smooth complete intersection curves. We also use our algorithm to close an open case left in a recent work of L. Costa, P. Macias Marques and R. M. Miró-Roig regarding the stability of the syzygy bundle of general forms. Finally, we apply our algorithm to provide a computational approach to tight closure. All algorithms are implemented in the computer algebra system CoCoA.

Mathematical Subject Classification (2010): primary: 14J60, 14Q15; secondary: 13P10

Keywords: Semistable vector bundle, syzygy bundle, Tannaka duality, monodromy group, tight closure

1. INTRODUCTION

The notion of slope-(semi)stability for vector bundles on a smooth projective varieties over an algebraically closed field K , as introduced by D. Mumford in the case of curves and generalized by F. Takemoto to higher dimensional varieties, is a very important tool in algebraic geometry. Unfortunately, for a concretely given vector bundle it is often very difficult to decide whether it is semistable or even stable. In this paper we develop an algorithm to determine computationally the semistability of certain vector bundles on the projective space \mathbb{P}^N . Throughout this paper we assume that $N \geq 2$, since for $N = 1$ by the Theorem of A. Grothendieck every vector bundle splits as a direct sum of line bundles. We restrict ourselves to vector bundles which are given as a kernel of a surjective morphism between splitting bundles, i.e., vector bundles \mathcal{E} which sit in a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i) \xrightarrow{\varphi} \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0.$$

We call such bundles *kernel bundles*. For instance, by the theorem of Horrocks every non-split vector bundle on \mathbb{P}^2 admits such a presentation. The morphism φ which defines \mathcal{E} is given by an $m \times n$ matrix $\mathcal{M} = (a_{ji})$, where the entries $a_{ji} \in R := K[X_0, \dots, X_N]$ are homogeneous polynomials of degrees $b_j - a_j$. Special instances ($m = 1$ and $b_1 = 0$) of kernel bundles are the so-called *syzygy bundles* $\text{Syz}(f_1, \dots, f_n)$ for R_+ -primary homogeneous polynomials f_1, \dots, f_n (i.e., $\sqrt{(f_1, \dots, f_n)} = R_+$), i.e., a syzygy bundle has a presenting sequence

$$0 \longrightarrow \text{Syz}(f_1, \dots, f_n) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(-d_i) \xrightarrow{f_1, \dots, f_n} \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0,$$

where $d_i = \deg(f_i)$. Due to their explicit nature, kernel bundles and syzygy bundles are suitable for direct computations, in particular Gröbner basis methods and combinatorics. But still in general, not much is known about (semi)stability of kernel bundles or even syzygy bundles. One of the most important results in this direction, due to H. Brenner, is a combinatorial criterion for (semi)stability of syzygy bundles given by monomial families:

Theorem 1.1 (Brenner). *Let K be a field, $R := K[X_0, \dots, X_N]$ and let $f_i = X^{\sigma_i}$ denote R_+ -primary monomials of degrees $d_i = |\sigma_i|$ in $K[X_0, \dots, X_N]$, $i = 1, \dots, n$. Suppose that for every subset $J \subseteq I := \{1, \dots, n\}$, $|J| \geq 2$, the inequality*

$$\frac{d_J - \sum_{i \in J} d_i}{|J| - 1} \leq \frac{-\sum_{i \in I} d_i}{n - 1}$$

holds, where d_J is the degree of the highest common factor of f_i , $i \in J$. Then the syzygy bundle $\text{Syz}(f_1, \dots, f_n)$ is semistable (and stable if $<$ holds).

Proof. See [7, Corollary 6.4]. \square

Another important theorem, due to G. Bohnhorst and H. Spindler, is a numerical (semi)stability criterion for kernel bundles of rank N on \mathbb{P}^N in characteristic 0:

Theorem 1.2 (Bohnhorst-Spindler). *Let \mathcal{E} be a vector bundle of rank $N \geq 2$ on the projective space \mathbb{P}^N over an algebraically closed field K of characteristic 0. Suppose there is a short exact sequence*

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{N+k} \mathcal{O}_{\mathbb{P}^N}(a_i) \longrightarrow \bigoplus_{j=1}^k \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0,$$

such that $a_1 \geq \dots \geq a_{N+k}$, $b_1 \geq \dots \geq b_k$ and $b_j > a_j$ for $j = 1, \dots, k$. Then \mathcal{E} is semistable (stable) if and only if $a_{N+k} \geq (>) \mu(\mathcal{E}) = \frac{1}{N}(\sum_{i=1}^{N+k} a_i - \sum_{j=1}^k b_j)$.

Proof. This is [3, Theorem 2.7] applied to the dual bundle \mathcal{E}^* . \square

A general algorithm using Gröbner bases methods (computation of syzygy modules) that detects semistability of syzygy bundles and its implementation

by the first author was already announced by H. Brenner in [7, Remark 5.3]. In this article we describe this algorithm more generally for kernel bundles and describe in detail how to implement it in a computer algebra system (this has been done concretely by the first author in CoCoA [12]). This semistability algorithm can be used as a tool to examine further problems regarding semi(stability) of vector bundles by providing interesting examples. We explain these applications in more detail in the sequel. The paper is organized as follows.

In Section 2 we recall a criterion due to H. J. Hoppe (see Lemma 2.2) which relates (semi)stability to global sections of exterior powers of a given vector bundle. In particular, we show that this result, originally only formulated in characteristic 0, holds in arbitrary characteristic. Hoppe's criterion is the key-result for our algorithm.

In Section 3 we discuss some properties of kernel bundles and syzygy bundles on projective spaces. In particular, for these bundles we discuss necessary Bohnhorst/Spindler like numerical conditions (compare Theorem 1.2) for semistability.

The actual semistability algorithm for kernel bundles and its implementation is explained in Section 4. Besides exterior powers, we also describe explicitly how to compute global sections of tensor products and symmetric powers of kernel bundles. These algorithms play an important role in our first application: the computation of Tannaka dual groups of polystable vector bundles \mathcal{E} of degree 0 and rank r on \mathbb{P}^N in characteristic 0.

Section 5 starts with a brief introduction to Tannaka duality. Roughly spoken, for a polystable vector bundle \mathcal{E} of degree 0 one can find a semisimple algebraic group $G_{\mathcal{E}}$ and an equivalence of categories between the abelian tensor category generated by \mathcal{E} and the category of finite-dimensional representations of $G_{\mathcal{E}}$. The algebraic group $G_{\mathcal{E}}$ is called the *Tannaka dual group* of \mathcal{E} . It was shown by the second author in [30, Lemma 4.4 and Proposition 5.3] that for stable vector bundles of degree 0 as in Theorem 1.2 the almost simple components of the Tannaka dual group are of type A . We explain how to compute the Tannaka dual group for an arbitrary stable kernel bundle of degree 0 on \mathbb{P}^N and construct examples for low-rank syzygy bundles on \mathbb{P}^2 having the symplectic group \mathbf{Sp}_r as Tannaka dual group.

Furthermore, we are interested in the behaviour of the Tannaka dual group after restricting the bundles to smooth curves. Section 6 contains a short overview of restriction theorems for sheaves.

In Section 7, we describe a method to construct for certain kernel bundles \mathcal{E} on \mathbb{P}^N a finite morphism $f : \mathbb{P}^N \rightarrow \mathbb{P}^N$ such that the restriction of the pull-back $f^*(\mathcal{E})$ to certain complete intersection curves of sufficiently large degree has the same Tannaka dual group as the vector bundle $f^*(\mathcal{E})$ on \mathbb{P}^N . We show

that this works for example for the syzygy bundles constructed in Section 5. We would like to draw the reader's attention to the paper [1] by V. Balaji. He shows the existence of a rank 2 bundle \mathcal{E} with $c_2(\mathcal{E}) \gg 0$ on a smooth surface X , such that the restriction to a curve of genus > 1 has Tannaka dual group \mathbf{SL}_2 , see also [2, Proposition 3]. His method is completely different from ours. He uses this result to show that the moduli space of stable principal H -bundles on X with large characteristic classes is non-empty, where H is any semisimple algebraic group ([1, Chapter 7]).

In Section 8 we close an open case left in the paper [13], where L. Costa, R. M. Miró-Roig and P. Macias Marques show the stability of the generic syzygy bundle on \mathbb{P}^2 except for the bundle generated by five generic quadrics. We use the results obtained in Section 5 and construct an example for a stable syzygy bundle in this case, which gives the generic result via the openness of stability.

In the final section we provide another application of the semistability algorithm concerning the computation of tight/solid closure of homogeneous ideals in the coordinate ring of a smooth projective curve. This is possible due to the geometric approach to this topic developed by H. Brenner.

Acknowledgement. We would like to thank Holger Brenner for many useful discussions. In particular the first author is grateful for the supervision of his PhD-thesis [29] at the University of Sheffield where the semistability algorithm is part of one chapter.

2. THEORETICAL BACKGROUND – HOPPE'S SEMISTABILITY CRITERION

We recall that a torsion-free sheaf \mathcal{E} on a smooth projective variety X over an algebraically closed field K is *semistable* if for every coherent subsheaf $0 \neq \mathcal{F} \subset \mathcal{E}$ the inequality $\mu(\mathcal{F}) := \deg(\mathcal{F})/\mathrm{rk}(\mathcal{F}) \leq \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E}) = \mu(\mathcal{E})$ holds. The sheaf \mathcal{E} is *stable* if the inequality is always strict. The degree of a sheaf \mathcal{F} is defined using intersection theory and a fixed very ample invertible sheaf $\mathcal{O}_X(1)$ (which is also called a *polarization* of X) as $\deg(\mathcal{F}) = \deg(c_1(\mathcal{F}) \cdot \mathcal{O}_X(1)^{\dim(X)-1})$. For every coherent torsion-free sheaf \mathcal{E} there exists a unique filtration $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_t = \mathcal{E}$, called the *Harder-Narasimhan filtration*, such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable and $\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \dots > \mu(\mathcal{E}/\mathcal{E}_{t-1})$. The slopes $\mu(\mathcal{E}_1)$ and $\mu(\mathcal{E}/\mathcal{E}_{t-1})$ are also denoted by $\mu_{\max}(\mathcal{E})$ and $\mu_{\min}(\mathcal{E})$ respectively. If K is not algebraically closed, then we define the terms degree, semistable, etc. via the algebraic closure of K .

If the characteristic of the base field K is 0, it is well-known that the tensor product $\mathcal{E} \otimes \mathcal{F}$ of two semistable vector bundles \mathcal{E} and \mathcal{F} on a smooth projective polarized variety $(X, \mathcal{O}_X(1))$ is again semistable, and this also holds for exterior powers and symmetric powers (cf. [27, Theorem 3.1.4 and Corollary 3.2.10]). This does not longer hold in characteristic $p > 0$. This is due to the fact that the (absolute) *Frobenius morphism* $F : X \rightarrow X$ may destroy

semistability, i.e., the Frobenius pull-back $F^*(\mathcal{E})$ of a semistable vector bundle \mathcal{E} is in general not semistable; see for instance the example of Serre in [21, Example 3.2]. But for vector bundles on a projective space (in which we are mainly interested in) semistability behaves nicely with respect to tensor operations.

Lemma 2.1. *Let $(X, \mathcal{O}_X(1))$ be a smooth projective polarized variety defined over an algebraically closed field K of positive characteristic such that $\mu_{\max}(\Omega_X) \leq 0$. If \mathcal{E} and \mathcal{F} are semistable vector bundles, then $\mathcal{E} \otimes \mathcal{F}$, all exterior powers $\bigwedge^q \mathcal{E}$ and all symmetric powers $S^q \mathcal{E}$ are semistable. In particular, this holds if X is an abelian variety, a toric variety or a homogeneous space.*

Proof. Since $\mu_{\max}(\Omega_X) \leq 0$ a semistable vector bundle \mathcal{E} on X is *strongly semistable* by [38, Theorem 2.1] (we recall that this means that the Frobenius pull-backs $F^{e*}(\mathcal{E})$ are semistable for all $e \geq 0$). Hence it follows from [44, Theorem 3.23] that $\mathcal{E} \otimes \mathcal{F}$, $\bigwedge^q \mathcal{E}$ and $S^q \mathcal{E}$ are also semistable.

The cotangent bundle of an abelian variety is trivial and the cotangent bundles of toric varieties and homogeneous spaces can be embedded into a trivial bundle. So these varieties fulfill the condition $\mu_{\max}(\Omega_X) \leq 0$ which gives the supplement. \square

The following result is well-known in characteristic 0 (see for instance [7, Proposition 2.1.5] or [3, Proposition 1.1]). It gives an algorithmic criterion to check semistability of a vector bundle on \mathbb{P}^N in terms of global sections of its exterior powers. It uses the trivial but useful fact (in particular for a computational approach to semistability) that a semistable vector bundle of negative degree (or slope) does not have any nontrivial global sections. Since the key-idea goes already back to H. J. Hoppe (see [25, Lemma 2.6]), this result is attributed to him. Lemma 2.1 shows that Hoppe's result is also true in positive characteristic. We mention that the proof which we present below is essentially the same as the one given in [7].

Proposition 2.2 (Hoppe). *Let \mathcal{E} be a vector bundle on \mathbb{P}^N over an algebraically closed field K . Then the following holds.*

- (1) *The bundle \mathcal{E} is semistable if and only if for every $q < \text{rk}(\mathcal{E})$ and every $k < -q\mu(\mathcal{E})$ there does not exist a non-trivial global section of $(\bigwedge^q \mathcal{E})(k)$.*
- (2) *If $\Gamma(\mathbb{P}^N, (\bigwedge^q \mathcal{E})(k)) = 0$ for every $q < \text{rk}(\mathcal{E})$ and every $k \leq -q\mu(\mathcal{E})$, then \mathcal{E} is stable.*

Proof. We prove (1). If \mathcal{E} is semistable, then all exterior powers $\bigwedge^q \mathcal{E}$ are also semistable by Lemma 2.1. We have $\mu(\bigwedge^q \mathcal{E}) = q\mu(\mathcal{E})$, which can be easily verified using the splitting principle (see [41, Section I.1.2]). We have

$\Gamma(\mathbb{P}^N, (\bigwedge^q \mathcal{E})(k)) = 0$ because

$$\begin{aligned} \mu((\bigwedge^q \mathcal{E})(k)) &= \mu((\bigwedge^q \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^N}(k)) \\ &= \mu(\bigwedge^q \mathcal{E}) + \mu(\mathcal{O}_{\mathbb{P}^N}(k)) \\ &= q\mu(\mathcal{E}) + k \\ &< 0 \end{aligned}$$

and \mathcal{E} is semistable.

For the proof of the other direction, we do not need Lemma 2.1. Assume that for every $q < \text{rk}(\mathcal{E})$ and every $k < -q\mu(\mathcal{E})$ there does not exist a non-trivial global section of $(\bigwedge^q \mathcal{E})(k)$. Let $\mathcal{F} \subset \mathcal{E}$ be a coherent subsheaf of rank $q < \text{rk}(\mathcal{E})$. Then we also have an inclusion $\bigwedge^q \mathcal{F} \subset \bigwedge^q \mathcal{E}$. The bidual $(\bigwedge^q \mathcal{F})^{**}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^N}(m)$ with $m = \deg(\mathcal{F})$. Hence, we have $\bigwedge^q \mathcal{F} \cong \mathcal{O}_{\mathbb{P}^N}(m)$ outside a closed subset of codimension ≥ 2 (see [41, Lemma 1.1.10]). Because \mathcal{E} is locally free, there is a non-trivial sheaf morphism $\mathcal{O}_{\mathbb{P}^N}(m) \rightarrow \bigwedge^q \mathcal{E}$, i.e., $\Gamma(\mathbb{P}^N, (\bigwedge^q \mathcal{E})(-m)) \neq 0$. By assumption we have $-m \geq -q\mu(\mathcal{E})$ and therefore $\mu(\mathcal{F}) = m/q \leq \mu(\mathcal{E})$. Hence the vector bundle \mathcal{E} is semistable.

Part (2) follows in the same way if we replace $<$ by \leq appropriately. \square

Remark 2.3. We recall that the concepts of semistability and stability coincide if $\deg(\mathcal{E})$ and $\text{rk}(\mathcal{E})$ are coprime. As mentioned in [3] the converse of the stability statement in Lemma 2.2 does not hold in general. The easiest examples are the so-called *nullcorrelation bundles* on projective spaces \mathbb{P}^N for N odd. These bundles are given by a short exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{T}_{\mathbb{P}^N}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^N}(1) \longrightarrow 0,$$

where $\mathcal{T}_{\mathbb{P}^N}$ denotes the tangent bundle. In particular, we have $\text{rk}(\mathcal{N}) = N - 1$ and $\deg(\mathcal{N}) = 0$. Moreover, the nullcorrelation bundles are stable and have the property that if $N \geq 5$, then $\Gamma(\mathbb{P}^N, \bigwedge^2 \mathcal{N}) \neq 0$ (see [ibid., Remark below Example 1.2]). So these bundles do not fulfill the exterior power condition from Lemma 2.2.

Nevertheless, it is easy to see that a rank-2 vector bundle \mathcal{E} on \mathbb{P}^N (N arbitrary) is stable if and only if $\Gamma(\mathbb{P}^N, \mathcal{E}(k)) = 0$ for $k \leq \mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{2}$. Since $\Gamma(\mathbb{P}^3, \mathcal{N}) = 0$ (see [41, Proof of Theorem II.1.3.1(i)]), it is clear that a nullcorrelation bundle \mathcal{N} on \mathbb{P}^3 is stable and fulfills the exterior power condition.

Remark 2.4. Let $(X, \mathcal{O}_X(1))$ be a polarized smooth projective variety of dimension $d \geq 1$ defined over an algebraically closed field of characteristic 0. For a semistable vector bundle \mathcal{E} on X the numerical condition on the exterior powers $\bigwedge^q \mathcal{E}$ in Lemma 2.2 is still fulfilled, if we replace the degree bound for the global sections by $k < -q\mu(\mathcal{E})/\deg(\mathcal{O}_X(1))$ for every $1 \leq q < \text{rk}(\mathcal{E})$.

Hence, the numerical condition is, up to the factor $1/\deg(\mathcal{O}_X(1))$, always necessary for semistability.

If additionally $\text{Pic}(X) = \mathbb{Z}$, then the numerical criterion is again equivalent to the semistability of \mathcal{E} . Important examples of varieties with this property are general surfaces of degree ≥ 4 in $\mathbb{P}_{\mathbb{C}}^3$ (Noether's Theorem) and (in arbitrary characteristic) complete intersections of dimension ≥ 3 in \mathbb{P}^N (see [20, Corollary IV.3.2 and IV.4(i)]).

In positive characteristic, (under the assumption $\text{Pic}(X) = \mathbb{Z}$) the numerical condition on the exterior powers still implies semistability, but the equivalence in Lemma 2.2 only holds if every semistable vector bundle on X is strongly semistable (see Lemma 2.1). Thus it is clear that Lemma 2.2 does not provide an algorithmic tool to detect semistability of vector bundles on curves. For algorithmic methods to determine semistability and strong semistability of vector bundles over an algebraic curve in positive characteristic see [29, Chapter 3].

Example 2.5. Let $F \in \mathbb{Z}[X_0, \dots, X_N]$, $N \geq 4$, be a homogeneous polynomial of degree d such that the hypersurface $X := \text{Proj}(\mathbb{Q}[X_0, \dots, X_N]/(F))$ is smooth. By Remark 2.4 we have $\text{Pic}(X) \cong \mathbb{Z}$ and thus Hoppe's criterion 2.2 is applicable to determine semistability of vector bundles on X . Now we assume that $d = N + 1$. Then the canonical bundle $\omega_X \cong \mathcal{O}_X$ is trivial which implies the semistability of the cotangent bundle Ω_X (see [43, Theorem 3.1]). In particular, X is a Calabi-Yau variety. We consider X as the generic fiber \mathcal{X}_0 of the generically smooth projective morphism

$$\mathcal{X} := \text{Proj}(\mathbb{Z}[X_0, \dots, X_N]/(F)) \longrightarrow \text{Spec } \mathbb{Z}$$

of relative dimension $N - 2$. Up to finitely many exceptions, the special fiber \mathcal{X}_p over a prime number p is a smooth projective variety over the finite field \mathbb{F}_p with $\text{Pic}(\mathcal{X}_p) = \mathbb{Z}$. By the openness of semistability, the cotangent bundle $\Omega_{\mathcal{X}_p}$ of the special fiber \mathcal{X}_p is semistable too for almost all prime numbers p . Since $\deg(\Omega_{\mathcal{X}_p}) = 0$, every semistable vector bundle is strongly semistable on \mathcal{X}_p by [38, Theorem 2.1]. Thus, for $p \gg 0$ we can also use Lemma 2.2 to detect semistability of vector bundles on \mathcal{X}_p (in positive characteristic).

3. SYZGY BUNDLES AND KERNEL BUNDLES

In the sequel of this article, we restrict ourselves to vector bundles on \mathbb{P}^N , $N \geq 2$, which are kernels of surjective morphisms between splitting bundles, i.e., bundles sitting inside a short exact sequence of the form

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i) \xrightarrow{\varphi} \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0,$$

where $n \geq m$. The morphism φ is given by an $m \times n$ matrix $\mathcal{M} = (a_{ji})$, where the entries $a_{ji} \in R := K[X_0, \dots, X_N]$ are homogeneous polynomials of

degrees $b_j - a_i$. In this paper, we call such a vector bundle a *kernel bundle*. Special instances of kernel bundles are *syzygy bundles* which correspond to the case $m = 1$ and $b_1 = 0$, i.e., a syzygy bundle $\text{Syz}(f_1, \dots, f_n)$ is given by a short exact sequence

$$0 \longrightarrow \text{Syz}(f_1, \dots, f_n) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(-d_i) \xrightarrow{f_1, \dots, f_n} \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0,$$

where $f_1, \dots, f_n \in R = K[X_0, \dots, X_N]$ are homogeneous polynomials of degrees d_i , $i = 1, \dots, n$. If one of the polynomials is constant, the syzygy bundle $\text{Syz}(f_1, \dots, f_n)$ is obviously split. To exclude this case, one often demands that the ideal (f_1, \dots, f_n) is R_+ -primary, i.e., $\sqrt{(f_1, \dots, f_n)} = R_+ = (X_0, \dots, X_N)$. The most prominent example of a syzygy bundle is the *cotangent bundle* $\Omega_{\mathbb{P}^N} \cong \text{Syz}(X_0, \dots, X_N)$ of \mathbb{P}^N .

We can compute the topological invariants of a kernel bundle \mathcal{E} from the presenting short exact sequence above. We have

$$\text{rk}(\mathcal{E}) = n - m \text{ and } \deg(\mathcal{E}) = c_1(\mathcal{E}) = \sum_{i=1}^n a_i - \sum_{j=1}^m b_j,$$

and thus

$$\mu(\mathcal{E}) = \frac{1}{n - m} \left(\sum_{i=1}^n a_i - \sum_{j=1}^m b_j \right).$$

Since the Chern polynomial is multiplicative on short exact sequences, it is also easy to compute higher Chern classes of kernel bundles (see also Section 6).

If \mathcal{E} does not split as a direct sum of line bundles, then the dual bundle \mathcal{E}^* of a kernel bundle has homological dimension one, and therefore we obtain the inequality $\text{rk}(\mathcal{E}) \geq N$ (see [3, Corollary 1.7]).

Example 3.1. Every vector bundle \mathcal{E} on the projective plane \mathbb{P}^2 which does not split as a direct sum of line bundles has homological dimension 1. This is easy to see, since there exists a surjective sheaf morphism $\mathcal{F} \rightarrow \mathcal{E}$ for some splitting bundle $\mathcal{F} = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^2}(a_i)$, which is also surjective on global sections (for this standard argument see for instance [3, Lemma 1.5]). Thus, there exists a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0,$$

where the first cohomology $H^1(\mathbb{P}^2, \mathcal{K})$ of the kernel \mathcal{K} vanishes. But by the theorem of Horrocks (see [41, Theorem I.2.3.1]) this means that \mathcal{K} splits as a direct sum of line bundles. Since \mathcal{E} is semistable if and only if \mathcal{E}^* is semistable, we can dualize the short exact sequence and apply Algorithm 4.6 to the kernel

bundle \mathcal{E}^* . Hence, our semistability algorithm is applicable to every (non-split) vector bundle on \mathbb{P}^2 and to vector bundles of homological dimension 1 on \mathbb{P}^N in general.

In the sequel, we show that the twists a_1, \dots, a_n and b_1, \dots, b_m , which occur in the presenting sequence of a kernel bundle, have to fulfill a certain numerical condition which is necessary for semistability (stability). We remark that this condition is also necessary for the semistability (stability) of kernel bundles (and in particular of syzygy bundles) on arbitrary smooth projective varieties.

If \mathcal{E} is a vector bundle on \mathbb{P}^N of rank r and $\mathcal{F} \subset \mathcal{E}$ a subsheaf of rank $r-1$, then the quotient \mathcal{E}/\mathcal{F} is outside codimension 2 isomorphic to $\mathcal{O}_{\mathbb{P}^N}(\ell)$ for some $\ell \in \mathbb{Z}$. This is equivalent to a section $\mathcal{O}_{\mathbb{P}^N} \rightarrow \mathcal{E}^*(-\ell)$. For kernel bundles we are able to control such sections by an easy numerical condition. In particular, we can replace the condition on the global sections of the $(r-1)$ th exterior power in Hoppe's criterion 2.2 by this condition. Before we state the result, we recall that a resolution

$$\mathfrak{F}_\bullet : 0 \longrightarrow \mathcal{F}_d \longrightarrow \dots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0$$

of a vector bundle \mathcal{E} on \mathbb{P}^N with splitting bundles \mathcal{F}_i , $0 \leq i \leq d$, is *minimal* if the global evaluation

$$\begin{aligned} 0 &\longrightarrow \Gamma(\mathbb{P}^N, \mathcal{F}_d(m)) \longrightarrow \dots \longrightarrow \Gamma(\mathbb{P}^N, \mathcal{F}_1(m)) \\ &\longrightarrow \Gamma(\mathbb{P}^N, \mathcal{F}_0(m)) \longrightarrow \Gamma(\mathbb{P}^N, \mathcal{E}(m)) \longrightarrow 0 \end{aligned}$$

is exact too for every $m \in \mathbb{Z}$ and no line bundle can be omitted for two consecutive splitting bundles \mathcal{F}_i and \mathcal{F}_{i-1} , which is equivalent to say that there are no constant entries ($\neq 0$) in the matrices representing the differentials in the resolution (see [42, Section 7.2]).

Lemma 3.2. *Let \mathcal{E} be a vector bundle on \mathbb{P}^N , $N \geq 2$, sitting in a short exact sequence*

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i) \longrightarrow \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0,$$

where $a_1 \geq a_2 \geq \dots \geq a_n$. If $a_n \geq (>) \mu(\mathcal{E}) = \frac{1}{n-m}(\sum_{i=1}^n a_i - \sum_{j=1}^m b_j)$, then there are no mappings from \mathcal{E} to line bundles which contradict the semistability (stability) of \mathcal{E} . Moreover, if the dualized sequence is a minimal resolution for \mathcal{E}^* , then this numerical condition is necessary for semistability (stability).

Proof. We twist the short exact sequence presenting \mathcal{E} with $\mathcal{O}_{\mathbb{P}^N}(\ell)$ and look at the dual sequence

$$0 \longrightarrow \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(-\ell - b_j) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(-\ell - a_i) \longrightarrow (\mathcal{E}(\ell))^* \cong \mathcal{E}^*(-\ell) \longrightarrow 0.$$

Since $H^1(\mathbb{P}^N, \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(-\ell - b_j)) = 0$ for all $\ell \in \mathbb{Z}$, every global section of $(\mathcal{E}(\ell))^*$ comes from $\Gamma(\mathbb{P}^N, \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(-\ell - a_i))$. Consequently, for $\ell > -a_n$ there exists no non-trivial morphism $\mathcal{E}(\ell) \rightarrow \mathcal{O}_{\mathbb{P}^N}$. By assumption, we have $-a_n \leq \mu(\mathcal{E}^*)$. So we have $\Gamma(\mathbb{P}^N, \mathcal{E}^*(-\ell)) = 0$ for $\ell > \mu(\mathcal{E}^*)$. Hence there are no mappings to line bundles which contradict the semistability. Analogously, one obtains the corresponding statement for stability.

Next, we prove the supplement. Assume that $a_n < (\leq) \mu(\mathcal{E})$. It follows from the assumption on the resolution of \mathcal{E}^* that the mapping $\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^N}(a_n)$ is non-zero. But such a morphism does not exist for \mathcal{E} semistable or stable. \square

Remark 3.3. If \mathcal{E} is a rank-3 bundle (not necessarily a kernel bundle), then \mathcal{E} is semistable if and only if $\Gamma(\mathbb{P}^N, \mathcal{E}(\ell)) = 0$ for $\ell < -\mu(\mathcal{E})$ and $\Gamma(\mathbb{P}^N, \mathcal{E}^*(k)) = 0$ for $k < -\mu(\mathcal{E}^*)$ (see [41, Remark II.1.2.6]). So for kernel bundles of rank 3 on \mathbb{P}^2 (for \mathbb{P}^3 see Theorem 1.2) we only have to check global sections and the numerical condition of Proposition 3.2.

Remark 3.4. It is easy to check that the subsheaf

$$\ker\left(\bigoplus_{i \neq n} \mathcal{O}_{\mathbb{P}^N}(a_i) \rightarrow \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(b_j)\right)$$

destabilizes the kernel bundle \mathcal{E} if $a_n < \mu(\mathcal{E})$ and the resolution of \mathcal{E}^* is minimal.

Remark 3.5. In [3, Proposition 2.3], G. Bohnhorst and H. Spindler show that for a kernel bundle \mathcal{E} of rank N on \mathbb{P}^N , the corresponding resolution of \mathcal{E}^* is minimal if and only if $a_1 \geq \dots \geq a_{N+k}$, $b_1 \geq \dots \geq b_k$ and $b_j > a_j$ for $j = 1, \dots, k$ holds. Hence, in characteristic 0, Theorem 1.2 shows that for kernel bundles of this type the numerical condition of Lemma 3.2 is even sufficient for semistability (stability). For syzygy bundles, we give an easy characteristic free proof of Theorem 1.2.

Proposition 3.6. *Let K be a field and let $f_1, \dots, f_{N+1} \in R = K[X_0, \dots, X_N]$ be homogeneous parameters of degrees $1 \leq d_1 \leq \dots \leq d_{N+1}$. If $d_1 + \dots + d_N \geq (N-1)d_{N+1}$, then the syzygy bundle $\text{Syz}(f_1, \dots, f_{N+1})$ is semistable on \mathbb{P}^N . If the inequality is strict, then $\text{Syz}(f_1, \dots, f_{N+1})$ is a stable bundle.*

Proof. We use Lemma 2.2 to check the semistability of $\text{Syz}(f_1, \dots, f_{N+1})$. For a subset $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, N+1\}$ we use the notation $d_I = \sum_{j=1}^k d_{i_j}$. We consider the Koszul complex

$$0 \longrightarrow \mathcal{F}_{N+1} \longrightarrow \mathcal{F}_N \longrightarrow \dots \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0$$

on \mathbb{P}^N associated to the parameters f_1, \dots, f_{N+1} , where

$$\mathcal{F}_k := \bigoplus_{|I|=k} \mathcal{O}_{\mathbb{P}^N}(-d_I)$$

for $k = 1, \dots, N + 1$. We have to show for every $i < N$ and every

$$m < -i\mu(\text{Syz}(f_1, \dots, f_{N+1})) = \frac{i \sum_{j=1}^{N+1} d_j}{N}$$

that $\Gamma(\mathbb{P}^N, (\bigwedge^i \text{Syz}(f_1, \dots, f_{N+1}))(m)) = 0$. For every $1 \leq i < N$ we have a surjection

$$\mathcal{F}_{i+1} \longrightarrow \bigwedge^i \text{Syz}(f_1, \dots, f_n) \longrightarrow 0.$$

Since the Koszul complex of a regular sequence is also globally exact, every global section of $(\bigwedge^i \text{Syz}(f_1, \dots, f_n))(m)$ comes from the bundle $\mathcal{F}_{i+1}(m)$. We have $\Gamma(\mathbb{P}^N, \mathcal{F}_{i+1}(m)) = 0$ for $m < d_1 + d_2 + \dots + d_{i+1}$. The assumption $\sum_{i=1}^N d_i \geq (N-1)d_{N+1}$ implies

$$(N-i)(d_1 + \dots + d_{i+1}) \geq i(d_{i+2} + \dots + d_{N+1}),$$

for $1 \leq i \leq N-1$ (for this easy computation see [7, Corollary 2.4]). But this is equivalent to $N(d_1 + \dots + d_{i+1}) \geq i(d_1 + \dots + d_{N+1})$, $1 \leq i \leq N-1$, and we obtain the assertion. The supplement follows analogously. \square

4. THE SEMISTABILITY ALGORITHM AND ITS IMPLEMENTATION

The aim of this section is to use Hoppe's semistability criterion 2.2 to obtain a semistability algorithm for kernel bundles, which can be implemented in a computer algebra system. All implementations described in this section can be found on

<http://www2.math.uni-paderborn.de/people/ralf-kasprowitz/cocoa.html>.

The major advantage of kernel bundles compared to arbitrary vector bundles is that we can compute the global sections $\Gamma(\mathbb{P}^N, \bigwedge^q \mathcal{E})$ of their exterior powers in a way which is suitable for a computer algebra system. We do not claim that such a computational approach is impossible for other vector bundles, but at least it requires more technical effort. For a kernel bundle \mathcal{E} we give (probably well-known) presentations of the tensor operations (tensor powers, exterior powers and symmetric powers) as kernels of mappings between splitting bundles (but these mappings are in general not surjective). This enables us to compute the global sections of these vector bundles described by applying the left exact functor $\Gamma(\mathbb{P}^N, -)$. For our semistability algorithm we require such a presentation only for the exterior powers, but we will need the other tensor operations in Section 5.

Proposition 4.1. *Let \mathcal{E} be a vector bundle on \mathbb{P}^N , which sits in a short exact sequence*

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i) \xrightarrow{\mathcal{M}=(a_{ji})} \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0.$$

Set $I := \{1, \dots, n\}$, $J := \{1, \dots, m\}$ and $Q := \{1, \dots, q\}$ for a fixed $q \in \mathbb{N}$.

The q th tensor product of \mathcal{E} sits in the exact sequence

$$(4.1) \quad 0 \longrightarrow \mathcal{E}^{\otimes q} \longrightarrow \bigoplus_{\alpha \in I^q} \mathcal{O}_{\mathbb{P}^N} \left(\sum_{p \in Q} a_{\alpha_p} \right) \xrightarrow{\varphi_q} \bigoplus_{\substack{(\beta, j, p) \\ \beta \in I^{q-1}, j \in J, p \in Q}} \mathcal{O}_{\mathbb{P}^N} \left(\sum_{p \in Q - \{q\}} a_{\beta_p} + b_j \right),$$

where the map φ_q is given by

$$e_\alpha \longmapsto \sum_{\substack{(j, p) \\ j \in J, p \in Q}} a_{j, \alpha_p} e_{(\alpha^{(p)}, j, p)}.$$

Here $\alpha^{(p)}$ means the $(q-1)$ -tuple α without the p th element.

The q th exterior power of \mathcal{E} , $1 \leq q < n - m$, sits in the exact sequence

$$(4.2) \quad 0 \longrightarrow \bigwedge^q \mathcal{E} \longrightarrow \bigoplus_{A \subseteq I, |A|=q} \mathcal{O}_{\mathbb{P}^N} \left(\sum_{i \in A} a_i \right) \xrightarrow{\varphi_q} \bigoplus_{\substack{(B, j) \\ B \subseteq I, |B|=q-1, j \in J}} \mathcal{O}_{\mathbb{P}^N} \left(\left(\sum_{i \in B} a_i \right) + b_j \right),$$

where the map φ_q is given by

$$e_A \longmapsto \sum_{\substack{(i, j) \\ i \in A, j \in J}} \text{sign}(i, A) a_{ji} e_{(A - \{i\}, j)}$$

(the subset $A \subset I$ is supposed to have the induced order and

$$\text{sign}(i, A) = \begin{cases} -1, & \text{if } i \text{ is an even element in } A, \\ 1, & \text{if } i \text{ is an odd element in } A. \end{cases}$$

Let $\text{char}(K) \nmid q$. The q th symmetric power of \mathcal{E} sits in the exact sequence

$$(4.3) \quad 0 \longrightarrow S^q(E) \longrightarrow \bigoplus_{\substack{i_1 \leq \dots \leq i_q \\ i_k \in I}} \mathcal{O}_{\mathbb{P}^N} \left(\sum_{k \in Q} a_{i_k} \right) \xrightarrow{\varphi_q} \bigoplus_{\substack{i'_1 \leq \dots \leq i'_{q-1}, j \\ i'_k \in I, j \in J}} \mathcal{O}_{\mathbb{P}^N} \left(\sum_{k \in Q - \{q\}} a_{i'_k} + b_j \right),$$

where the map φ_q is given by

$$e_{i_1 \leq \dots \leq i_q} \longmapsto \sum_{\substack{i \in \{i_1, \dots, i_q\} \\ j \in J}} a_{ji} e_{i_1 \leq \dots \leq \hat{i} \leq \dots \leq i_q, j}.$$

Here \hat{i} means that this element is omitted.

Proof. We start with formula (4.1). It is obviously true for $q = 1$, and assume that we proved it for some $q \in \mathbb{N}$. We consider the locally free sheaves

$$C_q^0 := \bigoplus_{\alpha \in I^q} \mathcal{O}_{\mathbb{P}^N} \left(\sum_{p \in Q} a_{\alpha_p} \right), \quad C_q^1 := \bigoplus_{\substack{(\beta, j, p) \\ \beta \in I^{q-1}, j \in J, p \in Q}} \mathcal{O}_{\mathbb{P}^N} \left(\sum_{p \in Q - \{q\}} a_{\beta_p} + b_j \right)$$

and obtain a complex $C_q: \dots \rightarrow 0 \rightarrow C_q^0 \xrightarrow{\varphi_q} C_q^1 \rightarrow 0 \rightarrow \dots$. The complex $C_q \otimes C_1$ is by definition the complex

$$\dots \rightarrow 0 \rightarrow C_q^0 \otimes C_1^0 \xrightarrow{\varphi_q \otimes \text{id} \oplus \text{id} \otimes \varphi_1} C_q^1 \otimes C_1^0 \oplus C_q^0 \otimes C_1^1 \rightarrow C_q^1 \otimes C_1^1 \rightarrow 0 \rightarrow \dots$$

see for example [10, page 63ff]. Since the kernels and images of all morphisms occuring in the complex C_1 are locally free, we may apply the formula of Künneth ([10, Theorem VI.3.1]) and get a short exact sequence

$$\begin{aligned} 0 &\longrightarrow \bigoplus_{r+s=k} H_r(C_q) \otimes H_s(C_1) \longrightarrow H_k(C_q \otimes C_1) \\ &\longrightarrow \bigoplus_{r+s=k-1} \text{Tor}_1(H_r(C_q), H_s(C_1)) \longrightarrow 0, \end{aligned}$$

where the Tor sheaves on the right are trivial due to the fact that the sheaves $H_s(C_1)$ are locally free for all s . Observe that the kernel of the morphism φ_q is $\mathcal{E}^{\otimes q}$, hence we get with the Künneth formula

$$\ker(\varphi_q \otimes \text{id} \oplus \text{id} \otimes \varphi_1) = H_0(C_q \otimes C_1) \cong H_0(C_q) \otimes H_0(C_1) = \mathcal{E}^{\otimes q} \otimes \mathcal{E}.$$

Furthermore, it is not difficult to obtain the identity $\varphi_{q+1} = \varphi_q \otimes \text{id} \oplus \text{id} \otimes \varphi_1$ under the obvious identifications $C_q^0 \otimes C_1^0 \cong C_{q+1}^0$ and $C_q^1 \otimes C_1^0 \oplus C_q^0 \otimes C_1^1 \cong C_{q+1}^1$.

For formula (4.2), let $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \rightarrow 0$ be a short exact sequence of locally free sheaves. Consider the Koszul complex of the \mathcal{O} -algebra $S\mathcal{G}$, locally defined by the sequence of elements in $S\mathcal{G}$ consisting of a basis of \mathcal{F} , i.e., the complex

$$0 \longrightarrow \bigwedge^s \mathcal{F} \otimes S\mathcal{G} \xrightarrow{d_s} \bigwedge^{s-1} \mathcal{F} \otimes S\mathcal{G} \xrightarrow{d_{s-1}} \dots \longrightarrow \mathcal{F} \otimes S\mathcal{G} \xrightarrow{d_1} S\mathcal{G} \longrightarrow 0,$$

with $s = \text{rk}(\mathcal{F})$ and

$$d_j(f_{i_1} \wedge \dots \wedge f_{i_j} \otimes g) = \sum_{k=1}^j (-1)^{k-1} (f_{i_1} \wedge \dots \wedge \hat{f}_{i_k} \wedge \dots \wedge f_{i_j}) \otimes f_{i_k} \cdot g$$

(\hat{f}_{i_k} means that we omit the k -th factor). One easily checks that for every $1 \leq q < \text{rk}(\mathcal{E})$ this yields an exact complex

$$0 \longrightarrow \bigwedge^q \mathcal{E} \longrightarrow \bigwedge^q \mathcal{F} \xrightarrow{\varphi_q} \bigwedge^{q-1} \mathcal{F} \otimes \mathcal{G} \longrightarrow \dots \longrightarrow \mathcal{F} \otimes S^{q-1} \mathcal{G} \longrightarrow S^q \mathcal{G} \longrightarrow 0,$$

with $\varphi_q := d_q$. Applying this argument for the short exact sequence defining the kernel bundle \mathcal{E} together with the isomorphism $\bigwedge^r \bigoplus_{i \in I} \mathcal{O}_{\mathbb{P}^N}(a_i) \cong \bigoplus_{A \subseteq I, |A|=q} \mathcal{O}_{\mathbb{P}^N}(\sum_{i \in A} a_i)$ yields the claim.

The case (4.3) works analogously to the second one, but now we take the Koszul complex of $S\mathcal{F}$ locally with respect to the regular sequence consisting

of a basis of \mathcal{E} . This gives the exact complex

$$0 \rightarrow \bigwedge^r \mathcal{E} \otimes S^{q-r} \mathcal{F} \rightarrow \bigwedge^{r-1} \mathcal{E} \otimes S^{q-2} \mathcal{F} \rightarrow \dots \rightarrow \mathcal{E} \otimes S \mathcal{F} \rightarrow S \mathcal{F} \rightarrow S \mathcal{G} \rightarrow 0.$$

Applying this to the dual exact sequence that defines the kernel bundle \mathcal{E} and dualizing again yields the exact complex

$$0 \rightarrow S^q(\mathcal{E}^*)^* \rightarrow S^q\left(\bigoplus_{i \in I} \mathcal{O}_{\mathbb{P}^N}(a_i)\right) \xrightarrow{d_1^*} \bigoplus_{j \in J} \mathcal{O}_{\mathbb{P}^N}(b_j) \otimes S^{q-1}\left(\bigoplus_{i \in I} \mathcal{O}_{\mathbb{P}^N}(a_i)\right)$$

Furthermore, there are isomorphisms

$$S^q\left(\bigoplus_{i \in I} \mathcal{O}_{\mathbb{P}^N}(a_i)\right) \cong \bigoplus_{\substack{i_1 \leq \dots \leq i_q \\ i_k \in I}} \mathcal{O}_{\mathbb{P}^N}\left(\sum_{k \in Q} a_{i_k}\right)$$

and $S^q(\mathcal{E}^*) \cong S^q(\mathcal{E})^*$ if the characteristic of the field K does not divide q , see [45, Satz 86.12]. \square

Now, we describe the implementation of the semistability algorithm for kernel bundles.

Remark 4.2. Let \mathcal{E} be a vector bundle on \mathbb{P}^N . The twisted dual bundle $\mathcal{E}^*(n)$ is generated by global sections for $n \gg 0$ (see [22, Theorem 5.17]), i.e., we have a surjection $\mathcal{O}_{\mathbb{P}^N}(-n)^s \rightarrow \mathcal{E}^* \rightarrow 0$ for some s . Let \mathcal{K} be the kernel of this morphism. Taking duals yields a short exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^N}(n)^s \rightarrow \mathcal{K}^* \rightarrow 0$$

of vector bundles. As in the proof of Proposition 4.1, we derive for every $1 \leq q \leq \text{rk}(\mathcal{E}) - 1$ an exact sequence

$$0 \rightarrow \bigwedge^q \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^N}(qn)^{\binom{s}{q}} \xrightarrow{\varphi_q} \mathcal{O}_{\mathbb{P}^N}((q-1)n)^{\binom{s}{q-1}} \otimes \mathcal{K}^*.$$

Since we also have a surjection $\mathcal{O}_{\mathbb{P}^N}(-m)^t \rightarrow \mathcal{K} \rightarrow 0$ for $m \gg 0$ and some t , we obtain a diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \bigwedge^q \mathcal{E} & \longrightarrow & \mathcal{O}_{\mathbb{P}^N}(qn)^{\binom{s}{q}} & \xrightarrow{\varphi_q} & \mathcal{O}_{\mathbb{P}^N}((q-1)n)^{\binom{s}{q-1}} \otimes \mathcal{K}^* \\ & & & \searrow \bar{\varphi}_q & & & \downarrow \\ & & & & & & \mathcal{O}_{\mathbb{P}^N}((q-1)n)^{\binom{s}{q-1}} \otimes \mathcal{O}_{\mathbb{P}^N}(m)^t, \end{array}$$

where the maps φ_q and $\bar{\varphi}_q$ have the same kernel. In this sense, the algorithmic methods to determine semistability which we develop in this chapter, are applicable to every vector bundle on \mathbb{P}^N .

Remark 4.3. Let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a finitely generated graded R -module ($R = K[X_0, \dots, X_N]$). If we fix homogeneous generators g_1, \dots, g_n , we obtain a surjection

$$R^n \xrightarrow{f} M, \quad e_j \mapsto g_j, \quad j = 1, \dots, n.$$

From this map we can derive, for every positive integer $q \geq 1$, the well-known exact sequence

$$(\ker f) \otimes_R \bigwedge^{q-1} R^n \xrightarrow{\alpha} \bigwedge^q R^n \longrightarrow \bigwedge^q M \longrightarrow 0,$$

where the map α is given by

$$x \otimes (y_1 \wedge \dots \wedge y_{q-1}) \mapsto x \wedge y_1 \wedge \dots \wedge y_{q-1}$$

(see [45, §83, Aufgabe 26]). Fixing homogeneous generators of $\ker f$ gives a diagram

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \uparrow & & & & \\ & & (\ker f) \otimes_R \bigwedge^{q-1} R^n & \xrightarrow{\alpha} & \bigwedge^q R^n & \longrightarrow & \bigwedge^q M \longrightarrow 0 \\ & \uparrow & & \nearrow \bar{\alpha} & & & \\ R^n \otimes_R \bigwedge^{q-1} R^n & & & & & & \end{array}$$

In this way we obtain a presentation of $\bigwedge^q M$ as a cokernel of a map between free modules. Since all these mappings are graded, we have a corresponding sequence

$$\bigoplus_{i=1}^{\tilde{n}} \mathcal{O}_{\mathbb{P}^N}(a_i) \xrightarrow{\bar{\alpha}} \bigoplus_{j=1}^{\tilde{m}} \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow \bigwedge^q \widetilde{M} \longrightarrow 0,$$

of the associated coherent sheaves on \mathbb{P}^N (with suitable twists). But the map $\bigoplus_{j=1}^{\tilde{m}} R_{b_j} \rightarrow \Gamma(\mathbb{P}^N, \bigwedge^q \widetilde{M})$ is in general not surjective. Hence this sequence cannot be the basis of an algorithmic approach. If the depth of M is at least 2, then this map is surjective if and only if $\bigwedge^q(\Gamma(\mathbb{P}^N, \widetilde{M})) \rightarrow \Gamma(\mathbb{P}^N, \bigwedge^q \widetilde{M})$ is surjective (see [16, Theorem A4.1 and Theorem A4.3]). An illustrating example is the following.

Example 4.4. We consider the syzygy bundle $\mathcal{S} := \text{Syz}(X^3, Y^3, Z^3, XY^2Z^2)$ on $\mathbb{P}^2 = \text{Proj } K[X, Y, Z]$ (see also [7, Example 7.4]) and use Brenner's criterion 1.1. The slope of this bundle equals $-\frac{14}{3} \approx -4,66$. For the subsheaves of rank 1 coming from two monomials we have (we only list the combinations having

a common factor)

$$\begin{aligned}\mu(\text{Syz}(X^3, XY^2Z^2)) &= 1 - 8 = -7, \\ \mu(\text{Syz}(Y^3, XY^2Z^2)) &= 2 - 8 = -6 \text{ and} \\ \mu(\text{Syz}(Z^3, XY^2Z^2)) &= 2 - 8 = -6.\end{aligned}$$

Hence we see that the global sections of \mathcal{S} do not contradict the semistability. But the monomial subfamily X^3, Y^3, Z^3 yields the rank-2 subbundle $\text{Syz}(X^3, Y^3, Z^3) \subset \mathcal{S}$ of slope $-\frac{9}{2} = -4.5$. Thus, \mathcal{S} is not semistable with Harder-Narasimhan filtration

$$0 \longrightarrow \text{Syz}(X^3, Y^3, Z^3) \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-5) \longrightarrow 0.$$

Moreover, the mapping $\bigwedge^2(\Gamma(\mathbb{P}^2, \mathcal{S})) \rightarrow \Gamma(\mathbb{P}^2, \bigwedge^2 \mathcal{S})$ is not surjective.

Since we assume that a vector bundle and its exterior powers are given as kernels of morphisms between splitting bundles, we have to know how to compute the kernel of an R -linear map $R^n \rightarrow R^m$ between finitely generated free modules over the polynomial ring R . The answer is given by the following well-known lemma which shows that we can compute a minimal nontrivial global section with Gröbner bases.

Lemma 4.5. *Let $R = K[X_0, \dots, X_N]$ be the polynomial ring over a field K and let $\varphi : R^m \rightarrow R^n$ be an R -linear map. Denote by e_1, \dots, e_m the standard basis vectors of R^m . With the notation $w_j = \varphi(e_j)$, $j = 1, \dots, m$, we have*

$$\ker \varphi = \text{Syz}_R(w_1, \dots, w_m).$$

In other words, the kernel of φ is the (R) -syzygy module of the columns of the matrix which describes φ .

Proof. See [31, Proposition 3.3.1(a)]. □

So we have accumulated the necessary technical tools to formulate an algorithm, based on Hoppe's criterion 2.2, to determine semistability of a given kernel bundle on \mathbb{P}^N . Note that every instruction can be performed with any computer algebra system which is able to handle Gröbner bases calculations. In our case, we implemented it in CoCoA [12].

Algorithm 4.6 (Semistability of kernel bundles).

Input: Two lists $[a_1, \dots, a_n]$, $[b_1, \dots, b_m]$ and a homogeneous $m \times n$ matrix $\mathcal{M} = (a_{ji})$ with no constant polynomial entries $a_{ji} \neq 0$ of degrees $b_j - a_i$ defining a kernel bundle

$$0 \longrightarrow \mathcal{E} = \widetilde{\ker \mathcal{M}} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i) \xrightarrow{\mathcal{M}} \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0$$

with $a_1 \geq a_2 \geq \dots \geq a_n$ (as usual $\widetilde{\ker \mathcal{M}}$ denotes the sheaf associated to the graded R -module $\ker \mathcal{M}$).

Output: The decision whether \mathcal{E} is semistable in terms of a boolean value TRUE or FALSE respectively.

- (1) Compute the invariants $\text{rk}(\mathcal{E}) = n - m$, $\deg(\mathcal{E}) = \sum_{i=1}^n a_i - \sum_{j=1}^m b_j$ and $\mu(\mathcal{E}) = \frac{1}{n-m} \left(\sum_{i=1}^n a_i - \sum_{j=1}^m b_j \right)$.
- (2) If the slope condition $a_n \geq \mu(\mathcal{E})$ of Proposition 3.2 is fulfilled, then continue. Else return FALSE and terminate.
- (3) Set $q := 1$.
- (4) Construct the matrix \mathcal{M}_q which describes the map φ_q in Proposition 4.1.
- (5) Compute the syzygy module S_q of the columns of \mathcal{M}_q .
- (6) Compute the initial degree $\alpha_q := \min\{t : (S_q)_t \neq 0\}$ of the graded R -module S_q (i.e., α_q is the minimal twist ℓ such that $\Gamma(\mathbb{P}^N, (\bigwedge^q \mathcal{E})(\ell)) \neq 0$).
- (7) If $\alpha_q < -q\mu(\mathcal{E})$, then return FALSE and terminate. Else set $q := q + 1$ and continue.
- (8) If $q < \text{rk}(\mathcal{E}) - 1$, then go back to step (4). Else return TRUE and terminate.

Remark 4.7. We require the lists $[a_1, \dots, a_n]$ and $[b_1, \dots, b_m]$ as input data, since they define the degree of every entry a_{ji} in the matrix \mathcal{M} . This becomes clear if the matrix \mathcal{M} contains zeros (which is often the case if $m \geq 2$). An illustrating example is $\text{Syz}(f_1, f_2, f_3, 0)$ on \mathbb{P}^2 for R_+ -primary elements f_1, f_2, f_3 . Since

$$\text{Syz}(f_1, f_2, f_3, 0) \cong \text{Syz}(f_1, f_2, f_3) \oplus \mathcal{O}_{\mathbb{P}^2}(-d),$$

the semistability heavily depends on the degree d which we have allocated to 0.

Remark 4.8. One easily verifies that the map given by the matrix \mathcal{M} is surjective if and only if the ideal generated by all $m \times m$ minors of \mathcal{M} is R_+ -primary. The latter property can be checked computationally.

We give a first example of an application of our semistability algorithm.

Example 4.9. Let K be an arbitrary field. We consider the monomials $X^2, Y^2, XY, XZ, YZ \in R = K[X, Y, Z]$ and the corresponding sheaf of syzygies $\mathcal{S} := \text{Syz}(X^2, Y^2, XY, XZ, YZ)$. Is \mathcal{S} a semistable sheaf? Since the ideal generated by these monomials is not R_+ -primary, we can neither apply Theorem 1.1 nor (at first sight) Algorithm 4.6. We compute a resolution of \mathcal{S} (for instance with CoCoA), namely,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-4)^2 \xrightarrow{\mathcal{A}} \mathcal{O}_{\mathbb{P}^2}(-3)^6 \longrightarrow \mathcal{S} \longrightarrow 0,$$

where

$$\mathcal{A} = \begin{pmatrix} x & 0 \\ -y & 0 \\ -y & x \\ 0 & -y \\ -z & 0 \\ 0 & z \end{pmatrix}.$$

Since \mathcal{S} is a reflexive sheaf, it is locally free on \mathbb{P}^2 (cf. [41, Lemma 1.1.10]). So if we dualize the resolution, we obtain a short exact sequence

$$0 \longrightarrow \mathcal{S}^* \longrightarrow \mathcal{O}_{\mathbb{P}^2}(3)^6 \xrightarrow{\mathcal{A}^t} \mathcal{O}_{\mathbb{P}^2}(4)^2 \longrightarrow 0,$$

i.e., \mathcal{S}^* is a kernel bundle. Hence, we apply Algorithm 4.6 to \mathcal{S}^* in order to obtain the answer to our question (we recall that \mathcal{S} is semistable if and only if \mathcal{S}^* is semistable). A CoCoA computation gives:

- (1) $H^0(\mathbb{P}^2, \mathcal{S}^*(m)) = 0$ for $m < -2$ and $-2 \geq -\mu(\mathcal{S}^*) = -\frac{5}{2}$.
- (2) $H^0(\mathbb{P}^2, (\wedge^2(\mathcal{S}^*))(m)) = 0$ for $m < -5$ and $-5 = -2\mu(\mathcal{S}^*)$. In particular, we obtain no information about stability.
- (3) The numerical condition of Proposition 3.2 is fulfilled. So there are no mappings $\mathcal{S}^* \rightarrow \mathcal{O}_{\mathbb{P}^2}(k)$ into line bundles which contradict the semistability.

Eventually, we conclude via Lemma 2.2 that \mathcal{S}^* is semistable and so is \mathcal{S} .

5. TANNAKA DUALITY OF STABLE SYZYGY BUNDLES

As a first application of the algorithms described in the previous sections we will compute the Tannaka dual groups of certain stable syzygy bundles of degree 0 on the projective plane. We start with describing the setting. From now on, K denotes an algebraically closed field of characteristic 0 and X a smooth, irreducible and projective variety over K . Furthermore, denote by \mathfrak{B}_X the category of polystable vector bundles of degree 0 on X . It is well-known that \mathfrak{B}_X is an abelian tensor category that possesses the faithful fiber functor $\omega_x : \mathfrak{B}_X \rightarrow \text{Vect}(K)$, where $\text{Vect}(K)$ is the category of finite-dimensional K -vector spaces and ω_x maps a bundle E to its fiber \mathcal{E}_x for a point $x \in X(K)$. In other words, it is a neutral Tannaka category and hence there exists an affine group scheme G_X over K and an equivalence of categories

$$\mathfrak{B}_X \xrightarrow{\sim} \mathbf{Rep}_{G_X}(K).$$

For the theory of Tannaka categories, see e.g. [14]. We denote by $\mathfrak{B}_{\mathcal{E}}$ the smallest Tannaka subcategory of \mathfrak{B}_X containing the vector bundle \mathcal{E} and by $G_{\mathcal{E}}$ its Tannaka dual group. The group scheme G_X is pro-reductive and $G_{\mathcal{E}}$ is a reductive linear algebraic group (not necessarily connected). It is in a natural way an algebraic subgroup of $\text{GL}_{\mathcal{E}_x}$. Furthermore, there is a faithfully flat morphism $G_X \rightarrow G_{\mathcal{E}}$. Since global sections of vector bundles

in \mathfrak{B}_X correspond to $G_{\mathcal{E}}$ -invariant elements of the fiber, it follows from [14, Proposition 3.1] that the algebraic group $G_{\mathcal{E}}$ is uniquely determined by the global sections of $T^{r,s}(\mathcal{E}) := \mathcal{E}^{\otimes r} \otimes (\mathcal{E}^*)^{\otimes s}$ for $r, s \in \mathbb{N}$. It even suffices to know the global sections of $\mathcal{E}^{\otimes r}$ for $r \in \mathbb{N}$, since the dual of a stable bundle \mathcal{E} occurs as a direct summand in some tensor power of \mathcal{E} .

Remark 5.1. The restriction to fields of characteristic 0 is essential, as the following example shows. It was communicated to us by H. Brenner. Let K be a field of positive characteristic p , $p \geq 3$. Consider the plane curve $C = V_+(X^{3p-1} + Y^{3p-1} + Z^{3p-1} + X^p Z^{2p-1})$. This curve is smooth by the Jacobian criterion. Now we look at the syzygy bundle $\mathcal{E} := \text{Syz}(X^2, Y^2, Z^2)(3)$ on C of degree 0. Since \mathcal{E} is stable on \mathbb{P}^2 by Proposition 3.6 and $p \geq 3$, it remains stable on C by Langer's restriction theorem 6.2. The Frobenius pull-back $F^*(\mathcal{E}) \cong \text{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$ has the non-trivial section $s := (ZX^{p-1}, ZY^{p-1}, Z^p + X^p)$ because we have the equation

$$\begin{aligned} X^{2p} \cdot ZX^{p-1} + Y^{2p} \cdot ZY^{p-1} + Z^{2p} \cdot (Z^p + X^p) &= \\ Z(X^{3p-1} + Y^{3p-1} + Z^{3p-1} + X^p Z^{2p-1}) &= 0 \end{aligned}$$

on the curve. It is easy to see that s has no zeros on C and that there is no further non-trivial section of $F^*(\mathcal{E})$. Hence $F^*(\mathcal{E})$ is a non-trivial extension of the structure sheaf by itself and therefore not polystable. Since $F^*(\mathcal{S}) \subset S^p(\mathcal{E})$, it follows that $S^p\mathcal{E}$ is not polystable either. The same holds for $\mathcal{E}^{\otimes p}$ since $S^p\mathcal{E}$ is a quotient of the p -fold tensor product. So we see that \mathfrak{B}_C is not a tensor-category and in particular not Tannakian.

Let us now consider stable bundles \mathcal{E} of degree 0 on the projective space \mathbb{P}^N .

Lemma 5.2. *The Tannaka dual group $G_{\mathcal{E}}$ of a stable vector bundle \mathcal{E} of degree 0 on \mathbb{P}^N is a connected semisimple group.*

Proof. Suppose that the algebraic group $G_{\mathcal{E}}$ is not connected. Then the representations of the finite quotient $G_{\mathcal{E}}/G_{\mathcal{E}}^0$ would correspond to a subcategory of $\mathfrak{B}_{\mathcal{E}}$ containing nontrivial finite vector bundles, see [40, Lemma 3.1]. But the latter form, together with the obvious fiber functor, a neutral Tannaka category (see [40, Proposition 3.7]), and the K -valued points of the Tannaka dual group are well known to coincide with the étale fundamental group if the characteristic of the ground field is 0. Hence there are no nontrivial finite vector bundles on the projective space. Furthermore, the reductive group $G_{\mathcal{E}}$ does not have any nontrivial characters due to the fact that $\text{Pic}(\mathbb{P}^N) = \mathbb{Z}$, hence it has to be semisimple. \square

It was shown in [30, Lemma 4.4 and Proposition 5.3] that for stable vector bundles of degree 0 and rank r as in Theorem 1.2 the almost simple components of the Tannaka dual group have to be of type A . We conjecture that it

is always the standard r -dimensional representation of \mathbf{SL}_r or its dual. One motivation for this paper was to construct examples of syzygy bundles on the projective space having a Tannaka dual group of type different from A . Note that one cannot simply try to guess an example for a syzygy bundle with group different from $\mathbf{SL}_r \subset \mathbf{GL}_r$, see the remark 5.8 below. Our idea is to exclude this case by constructing self-dual bundles. With the algorithmic methods described in 4.6 and Proposition 4.1, it is in principle possible to compute the Tannaka dual group and its representation for an arbitrary stable kernel bundle of degree 0. However, the necessary computations grow very fast with the rank of the bundle, so we were only able to handle syzygy bundles up to rank 6 on \mathbb{P}^2 . Furthermore, we only found syzygy bundles having the almost simple Tannaka dual group $\mathbf{Sp}_r \subset \mathbf{GL}_r$, where $r \in \{4, 6\}$. There are no stable self-dual syzygy bundles of odd rank on \mathbb{P}^2 , see Corollary 5.4.

For stable rank 2 bundles of degree 0 on \mathbb{P}^2 , there is only one possible Tannaka dual group, namely the 2-dimensional irreducible representation of \mathbf{SL}_2 . An example for this is the syzygy bundle $\text{Syz}(X^2, Y^2, Z^2)(3)$. It is stable due to Theorem 1.2. To find higher rank syzygy bundles whose Tannaka dual group is not the group \mathbf{SL}_r with an r -dimensional representation, we will use the following simple Lemma.

Lemma 5.3. *Let $f_1, \dots, f_n \in R := K[X, Y, Z]$ be homogeneous polynomials such that the ideal $I := (f_1, \dots, f_n)$ is R_+ -primary and minimally generated by f_1, \dots, f_n . Then $\mathcal{E} := \text{Syz}(f_1, \dots, f_n)$ is self-dual (up to a twist with a line bundle) if and only if R/I is Gorenstein.*

Proof. The minimal free resolution of R/I has length 3 and is self-dual up to twist since R/I is Gorenstein:

$$0 \longrightarrow R(-d) \xrightarrow{\varphi} \bigoplus_{i=1}^n R(-e_i) \longrightarrow \bigoplus_{i=1}^n R(-d_i) \xrightarrow{f_1, \dots, f_n} R \longrightarrow R/I \longrightarrow 0$$

and with $E := \ker(f_1, \dots, f_n)$ we have $\text{coker}(\varphi) = E(-d)^*$. In particular, there is an isomorphism $E \cong E(-d)^*$ with $\mathcal{E} = \tilde{E}$. Conversely, one easily sees that if \mathcal{E} is self-dual up to twist, then the minimal free resolution of R/I ends with a free module of rank 1, that is R/I is Gorenstein. \square

Corollary 5.4. *There are no self-dual (up to a twist with a line bundle) non-split syzygy bundles of odd rank on \mathbb{P}^2 .*

Proof. This is [9], Corollary 2.2, which says that the minimal number of generators of a Gorenstein ideal of grade 3 is odd. \square

Corollary 5.5. *All stable syzygy bundles of degree 0 and odd rank ≤ 11 on the projective plane have a semisimple Tannaka dual group whose simple components are of type A .*

Proof. A table of representations of Lie algebras (e.g. [37]) shows that the smallest non-self-dual irreducible representation of a semisimple algebraic group with some simple component not of type A is the (up to duality) 12-dimensional representation of the semisimple group $\mathbf{SL}_3 \oplus \mathbf{Sp}_4$, which is the tensor product of the 3-dimensional irreducible representation of \mathbf{SL}_3 with the 4-dimensional irreducible representation of \mathbf{Sp}_4 . \square

Now looking at rank 4, what possibilities are there for the Lie algebra of the Tannaka dual group? There are three different self-dual and irreducible representations, where we use the notations of the tables of simple Lie algebras and their representations in [37]: A_1 with highest weight 3, C_2 with highest weight $(1, 0)$, and $A_1 \oplus A_1$ with highest weight $(1, 1)$. Using the computer algebra package for Lie group computations LiE [19], one finds that $\dim(\Gamma(\mathbb{P}^2, \mathcal{E}^{\otimes 4})) = \dim((\mathcal{E}_x^{\otimes 4})^{G_{\mathcal{E}}}) = 4$ in the cases of type A and $\dim(\Gamma(\mathbb{P}^2, \mathcal{E}^{\otimes 4})) = 3$ in the case of C_2 . To find a bundle with Tannaka dual group of type C_2 , we have to construct a Gorenstein ideal I with 5 minimal generators, such that the associated syzygy bundle \mathcal{E} has degree 0. Then we have to check its stability and the global sections of $\mathcal{E}^{\otimes 4}$.

To find a suitable Gorenstein ideal I , we consider the polynomial ring $R := K[X, Y, Z]$ as a module over itself by interpreting a polynomial in R as differential operator acting on itself, e.g., $X \cdot f = \frac{\partial f}{\partial X}$. Choose a homogeneous polynomial f of degree r and define $I := \text{Ann}_R(f) \subseteq R$. This ideal is Artinian and Gorenstein, see for example [28], Lemma 2.12.

Example 5.6. The Gorenstein ideal of the homogeneous form

$$f := X^2 + Y^2 + Z^2$$

is the ideal $I = (X^2 - Y^2, X^2 - Z^2, XY, XZ, YZ)$. If we pull back the associated syzygy bundle on \mathbb{P}^2 via the finite morphism $X \mapsto X^2, Y \mapsto Y^2, Z \mapsto Z^2$, we obtain the twisted syzygy bundle

$$\mathcal{E}(5) := \text{Syz}(X^4 - Y^4, X^4 - Z^4, X^2Y^2, X^2Z^2, Y^2Z^2)(5).$$

It has degree 0, rank 4 and is self-dual by Lemma 5.3. We apply Algorithm 4.6 to \mathcal{E} and obtain with the help of CoCoA:

- (1) $H^0(\mathbb{P}^2, \mathcal{E}(m)) = 0$ for $m \leq 5 = -\mu(\mathcal{E})$;
- (2) $H^0(\mathbb{P}^2, (\bigwedge^2 \mathcal{E})(m)) = 0$ for $m < 10 = -2\mu(\mathcal{E})$, $H^0(\mathbb{P}^2, (\bigwedge^2 \mathcal{E})(m)) \neq 0$ for $m = 10$;
- (3) $H^0(\mathbb{P}^2, (\bigwedge^3 \mathcal{E})(m)) = 0$ for $m \leq 15 = -3\mu(\mathcal{E})$ (this computation is, by Proposition 3.2, actually not necessary since the degrees of the polynomials are constant).

Hence, we see that \mathcal{E} (and all its twists) are semistable, but we get no information about stability because of (2). Observe that Hoppe's criterion can never reveal stability of a self-dual bundle of rank 4, since in this case there must be nontrivial global sections of $\Lambda^2(\mathcal{E}(5))$, see again [37].

Fortunately, the self-duality of the bundle allows us to prove its stability. By the computation above, we only have to consider subsheaves of rank 2 which may destroy the stability of $\mathcal{E}(5)$. Assume that $\mathcal{F} \subset \mathcal{E}(5)$ is a stable subsheaf of rank 2 and degree 0. Since we can pass over to the reflexive hull, and since we are working on \mathbb{P}^2 , we may assume that \mathcal{F} is locally free (see [41, Lemma 1.1.10]). In particular, we have

$$\mathcal{F} \cong \mathcal{F}^* \otimes \det(\mathcal{F}) \cong \mathcal{F}^* \otimes \mathcal{O}_{\mathbb{P}^2} \cong \mathcal{F}^*$$

by [23, Proposition 1.10]. That is, the subsheaf \mathcal{F} is self-dual too. So the composition of the morphisms

$$\mathcal{E}(5) \cong \mathcal{E}(5)^* \longrightarrow \mathcal{F}^* \cong \mathcal{F} \hookrightarrow \mathcal{E}(5)$$

yields an endomorphism of $\mathcal{E}(5)$, which is not a multiple of the identity. But a computation of global sections using the implementation of Proposition 4.1 yields that

$$h^0(\mathbb{P}^2, \text{End}(\mathcal{E}(5))) = h^0(\mathbb{P}^2, \mathcal{E}(5) \otimes \mathcal{E}(5)) = 1,$$

that is, the bundle $\mathcal{E}(5)$ is simple and a morphism as above does not exist. Hence, the bundle $\mathcal{E}(5)$ is stable. Finally, another computation shows $h^0(\mathbb{P}^2, (\mathcal{E}(5)^{\otimes 4})) = 3$, hence the Tannaka dual group is in fact almost simple of type C_2 and the representation of its Lie algebra has highest weight $(1, 0)$. It is well-known that this corresponds to the irreducible and faithful representation $\mathbf{Sp}_4 \subset \mathbf{GL}_4$.

Example 5.7. The Gorenstein ideal associated to the homogeneous form $X^3Y + Y^3Z + Z^3X$ via the correspondence described above is the ideal

$$I := (X^3 - Y^2Z, Y^3 - XZ^2, X^2Y - Z^3, XY^2, YZ^2, X^2Z, XYZ).$$

We consider the same pull-back as in the previous example to be able to twist the associated syzygy bundle to degree 0. The same computations as above show that the corresponding syzygy bundle $\mathcal{E}(7)$, defined as

$$\text{Syz}(X^6 - Y^4Z^2, Y^6 - X^2Z^4, X^4Y^2 - Z^6, X^2Y^4, Y^2Z^4, X^4Z^2, X^2Y^2Z^2)(7)$$

is semistable of degree 0, where a subsheaf \mathcal{F} that destroys stability has to be of rank $r = 4$ or $r = 2$. But since $\mathcal{E}(7)$ is self-dual, we can always assume \mathcal{F} to be of rank 2 and obtain stability for the same reason as above, since the bundle again turns out to be simple. Again using LiE [19], we find that it is possible to determine the Lie algebra of the Tannaka dual group by computing $\dim(\mathcal{E}^{\otimes 4}) = 3$, which shows that it has to be simple of type C_3 , with highest weight $(1, 0, 0)$. This corresponds to the faithful irreducible representation $\mathbf{Sp}_6 \subset \mathbf{GL}_6$.

Remark 5.8. There is a good reason why one should expect to find the group \mathbf{Sp}_r in these cases. It is well-known that the moduli space of stable bundles of fixed rank and Chern classes exists as a quasi-projective variety, see for example [27, Theorem 4.3.4]. Let us denote by M the moduli space

containing the syzygy bundle of Example 5.6 respectively of Example 5.7. Let \mathcal{U} be the quasi-universal bundle on $\mathbb{P}^2 \times M$, see [27, Chapter 4.6]. This means that the restriction of \mathcal{U} to $\mathbb{P}^2 \times \{p\}$ is a finite product of copies of the stable bundle on \mathbb{P}^2 corresponding to the point p . Applying the semicontinuity theorem (e.g. [22, Theorem 12.8]) for $\dim(H^0(\mathbb{P}^2 \times \{p\}, \mathcal{U}^{\otimes r}))$ one finds that the locus of the vector bundles having Tannaka dual group \mathbf{SL}_r is open in M since the dimension of $\Gamma(\mathbb{P}^2 \times \{p\}, \mathcal{U}^{\otimes r})$ is minimal in this case and strictly higher in all other cases. Hence for a generic choice of a stable syzygy bundle on \mathbb{P}^2 one expects to find the group \mathbf{SL}_r as Tannaka dual group. The locus of self-dual bundles is closed in M (apply the semicontinuity theorem for $\mathcal{U} \otimes \mathcal{U}$), containing the bundles with Tannaka dual group \mathbf{Sp}_r as an open locus (since the dimension of $\Gamma(\mathbb{P}^2 \times \{p\}, \mathcal{U}^{\otimes 4})$ is minimal for a self-dual stable bundle with this Tannaka dual group, see the discussion of the examples above). It follows that for a generic choice of a self-dual syzygy bundle one expects the group \mathbf{Sp}_r as Tannaka dual group. It would be interesting to find a method for constructing stable syzygy bundles having a Tannaka dual group different from \mathbf{SL}_r or \mathbf{Sp}_r . Furthermore, one could try to determine the geometry of the Tannaka-strata of the moduli spaces discussed above.

6. RESTRICTION THEOREMS – A BRIEF OVERVIEW

Algorithm 4.6 enables us to determine semistability (and in some cases stability) of kernel bundles on projective spaces. If our algorithm gives a positive answer, then what can be said about the semistability (respectively stability) of these bundles when we restrict them to a hypersurface $X \subset \mathbb{P}^N$? The answer is given by so-called *restriction theorems* which ensure the semistability (stability) of the restriction of a bundle to hypersurfaces of sufficiently large degree. These theorems hold in more general situations, we only give formulations for vector bundles on projective spaces. By applying restriction theorems, we can use Algorithm 4.6 to produce examples of semistable vector bundles on more complicated projective varieties.

We omit the famous restriction theorem of Mehta and Ramanathan (see [39, Theorem 6.1] or [27, Theorem 7.2.8]), which works over an arbitrary algebraically closed field K and provides the existence of an integer k_0 such that for c general elements $D_1, \dots, D_c \in |\mathcal{O}_{\mathbb{P}^N}(k)|$ the restriction of a semistable torsion-free sheaf \mathcal{E} on \mathbb{P}^N to the smooth complete intersection $D_1 \cap \dots \cap D_c$ is semistable for all $k \geq k_0$. But their theorem provides no bound for k_0 . Unlike the theorem of Mehta and Ramanathan, the following restriction theorem of H. Flenner gives an explicit bound for k_0 , but only works in characteristic 0.

Theorem 6.1 (Flenner). *Let K be an algebraically closed field of characteristic 0 and let \mathcal{E} be a semistable coherent torsion-free sheaf of rank r on \mathbb{P}^N .*

Then for k and $1 \leq c \leq N - 1$ fulfilling

$$\frac{\binom{k+N}{N} - ck - 1}{k} > \max\left\{\frac{r^2 - 1}{4}, 1\right\}$$

and for c general elements $D_1, \dots, D_c \in |\mathcal{O}_{\mathbb{P}^N}(k)|$ the restriction $\mathcal{E}|_C$ is semi-stable on the smooth complete intersection $C = D_1 \cap \dots \cap D_c$.

Proof. See [18, Theorem 1.2] or [27, Theorem 7.1.1]. \square

The strongest restriction theorem is due to A. Langer. It works in arbitrary characteristic and gives a degree bound for arbitrary smooth hypersurfaces in projective space. It involves the *discriminant* $\Delta(\mathcal{E}) := 2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2$ of a locally free sheaf \mathcal{E} of rank r , where $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ denote the first and second Chern class respectively.

Theorem 6.2 (Langer). *Let K be an algebraically closed field and let \mathcal{E} be a stable coherent torsion-free sheaf of rank $r \geq 2$ on \mathbb{P}^N and let $D \in |\mathcal{O}_{\mathbb{P}^N}(k)|$ be a smooth divisor such that $\mathcal{E}|_D$ is torsion-free. If*

$$k > \frac{r-1}{r}\Delta(\mathcal{E}) + \frac{1}{r(r-1)},$$

then the restriction $\mathcal{E}|_D$ is stable.

Proof. See [33, Theorem 2.19]. \square

For a kernel bundle \mathcal{E} given by a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(a_i) \longrightarrow \bigoplus_{j=1}^m \mathcal{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0$$

we have already mentioned in Section 3 that $c_1(\mathcal{E}) = \sum_{i=1}^n a_i - \sum_{j=1}^m b_j$. Since the Chern polynomial is multiplicative on short exact sequences, it is also easy to see that we have

$$c_2(\mathcal{E}) = \frac{1}{2} \left(\left(\sum_{i=1}^n a_i \right)^2 - \sum_{i=1}^n a_i^2 + \left(\sum_{j=1}^m b_j \right)^2 + \sum_{j=1}^m b_j^2 \right) - \sum_{i,j} a_i b_j$$

and hence

$$\Delta(\mathcal{E}) = \left(\sum_{i=1}^n a_i \right)^2 + \left(\sum_{j=1}^m b_j \right)^2 - (n-m) \left(\sum_{j=1}^m b_j^2 - \sum_{i=1}^n a_i^2 \right) - 2 \sum_{i,j} a_i b_j.$$

In particular, we have for a syzygy bundle $\mathcal{S} := \text{Syz}(f_1, \dots, f_n)$ given by homogeneous R_+ -primary polynomials $f_1, \dots, f_n \in R = K[X_0, \dots, X_N]$ of degrees d_1, \dots, d_n the formula $\Delta(\mathcal{S}) = (\sum_{i=1}^n d_i)^2 - (n-1) \sum_{i=1}^n d_i^2$.

In [5] H. Brenner has shown that there is no restriction theorem for strong semistability in the sense of Theorem 6.2. For general hypersurfaces $X \subset \mathbb{P}^N$ there is the following Flenner-type restriction theorem which is also due to A. Langer.

Theorem 6.3 (Langer). *Let \mathcal{E} be a semistable locally free sheaf of rank $r \geq 2$ on \mathbb{P}^N over an algebraically closed field K of positive characteristic. Let k be an integer such that*

$$k > \frac{1}{2} \max \{ \Delta(\mathcal{E}), N^5 - 2N^3 + 2N + 1 \}$$

and

$$\frac{\binom{k+N}{N} - 1}{k} > \max \left\{ \frac{r^2 - 1}{4}, 1 \right\} + 1.$$

Then $\mathcal{E}|_D$ is strongly semistable on the general hypersurface in $|\mathcal{O}_{\mathbb{P}^N}(k)|$.

Proof. See [34, Theorem 3.1]. \square

7. TANNAKA DUALITY OF STABLE BUNDLES RESTRICTED TO CURVES

We return to the situation of section 5. Recall that the ground field K is algebraically closed and of characteristic 0. Here we investigate the problem of the behavior of Tannaka dual groups after restricting a stable bundle of degree 0 on \mathbb{P}^N to smooth connected curves X such that the restricted bundle is still stable. The main problem is that the connected Tannaka dual group of a stable vector bundle on \mathbb{P}^N may become disconnected after restriction to a curve:

Lemma 7.1. *Let $X \subset \mathbb{P}^2$ be a connected smooth curve of genus > 1 . Then there exists a stable bundle of degree 0 on \mathbb{P}^2 such that its restriction to X is again stable with nonconnected Tannaka dual group.*

Proof. Recall that a finite vector bundle is a vector bundle which is trivialized by a finite étale morphism. There is a one-to-one correspondence between finite vector bundles on X and representations of the étale fundamental group $\pi_1(X, x)$ having finite image. Now choose such an irreducible representation of dimension r with trivial determinant, which certainly exists if the genus of the curve is > 1 . It is well-known that the associated vector bundle E is stable of degree 0, with Tannaka dual group equal to the image of the representation. Further, its determinant is the trivial bundle. Since $E^*(n)$ is generated by $r+1$ global sections ([6, Lemma 2.3]), it is easy to see that one obtains a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(n) \xrightarrow{\varphi} \bigoplus_{i=1}^{r+1} \mathcal{O}_X(d_i) \longrightarrow E^*(n) \longrightarrow 0.$$

The dual of the morphism φ lifts to an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}^2}(-d_i) \xrightarrow{\varphi^*} \mathcal{O}_{\mathbb{P}^2}(-n)$$

on \mathbb{P}^2 . In particular, $\mathcal{E}|_X = E$ and the singularities of \mathcal{E} are of codimension > 2 , hence \mathcal{E} is locally free and of course stable of degree 0. The Tannaka dual group of \mathcal{E} is connected by Lemma 5.2, but the restriction of the bundle \mathcal{E} to X has a finite Tannaka dual group. \square

Hence we need a criterion for the Tannaka dual group to be connected after restricting the bundle \mathcal{E} to the smooth and connected curve X . For the rest of this section we denote by p a prime number and by $\overline{\mathbb{Q}_p}$ an algebraic closure of the p -adic numbers with ring of integers \mathfrak{o} and residue field $\kappa = \overline{\mathbb{F}_p}$. We call a finitely presented, flat and proper scheme \mathfrak{X} over \mathfrak{o} together with an isomorphism $X \cong \mathfrak{X} \otimes_{\mathfrak{o}} \overline{\mathbb{Q}_p}$ a *model* of X . Note that any scheme \mathfrak{X} over \mathfrak{o} is the disjoint union of the generic fiber $\mathfrak{X} \otimes \overline{\mathbb{Q}_p}$, which is open in \mathfrak{X} , and the special fiber $\mathfrak{X} \otimes \overline{\mathbb{F}_p}$, which is closed. For the rest of this section set $K = \overline{\mathbb{Q}_p}$.

Theorem 7.2. *Let E be a vector bundle on the smooth, connected and projective curve X over $\overline{\mathbb{Q}_p}$. If there exists a model \mathfrak{X} of X together with a vector bundle \mathcal{E} on \mathfrak{X} such that $E \cong \mathcal{E} \otimes_{\mathfrak{o}} \overline{\mathbb{Q}_p}$ and $\mathcal{E} \otimes_{\mathfrak{o}} \mathfrak{o}/p$ is a trivial bundle on the scheme $\mathfrak{X} \otimes_{\mathfrak{o}} \mathfrak{o}/p$, then E is semistable of degree 0 with connected Tannaka dual group G_E .*

Proof. The proof uses results from non-abelian p -adic Hodge theory. The semistability of the bundle is shown in [15, Theorem 13], the assertion that G_E is connected follows from [30, Theorem 3.12]. \square

Example 7.3. Let \mathcal{E} be a vector bundle on \mathbb{P}^N sitting in the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{N+1} \mathcal{O}_{\mathbb{P}^N}(1) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^N}(N+1) \longrightarrow 0,$$

where the morphism φ is defined by homogeneous polynomials

$$f_i := X_0^{i-1} X_1^{N-i+1} + p g_i, \quad i = 1, \dots, N+1,$$

with $g_i \in \mathfrak{o}[X_0, \dots, X_N]$. The vector bundle \mathcal{E} is stable of degree 0 due to Theorem 1.2. If we consider \mathcal{E} as a sheaf on $\mathbb{P}_{\mathfrak{o}}^N$, it is easy to see that it is a locally-free sheaf outside the closed subset $\{[0; 0; X_2; \dots; X_N]\} \subset \mathbb{P}_{\kappa}^N \subset \mathbb{P}_{\mathfrak{o}}^N$. Then for every smooth connected curve $X \subset \mathbb{P}^N$ which has a model $\mathfrak{X} \subset \mathbb{P}_{\mathfrak{o}}^N$ such that the special fiber does not intersect the subspace $\{[0; 0; X_2; \dots; X_N]\}$, the vector bundle $\mathcal{E}|_{\mathfrak{X}}$ has N linearly independent global sections modulo p and hence is trivial. If the curve X is the intersection of $N-1$ smooth divisors of degree $\gg 0$, Theorem 6.2 yields that the restriction E of \mathcal{E} to X is a stable bundle, and it follows from Theorem 7.2 above that G_E is connected. See also [30, Example 4.6 and Remark 4.8].

Remark 7.4. A similar argument was used by H. Brenner to provide examples for stable vector bundles on p -adic curves with semistable reduction, see [15, Remark p. 571].

In general, we cannot apply Theorem 7.2 to an arbitrary kernel bundle \mathcal{E} on \mathbb{P}^N , but in many cases one can apply it to the pull-back $\pi^*(\mathcal{E})$ with respect to a suitable chosen finite morphism $\pi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ such that $\pi^*(\mathcal{E})$ is defined over \mathfrak{o} and such that it is trivial modulo p outside a closed subset of codimension ≥ 2 . Then for all curves X which have a model $\mathfrak{X} \subset \mathbb{P}_{\mathfrak{o}}^N$ whose special fiber does not intersect this closed subset, we have $G_{\pi^*(\mathcal{E})} = G_{\pi^*(\mathcal{E})|_X}$ if the curve X is a complete intersection of smooth divisors of sufficiently high degree. We will illustrate this method in the following examples.

Example 7.5. Consider the syzygy bundles

$$\mathcal{E}_1 = \text{Syz}(X^4 - Y^4, X^4 - Z^4, X^2Y^2, X^2Z^2, Y^2Z^2)(5)$$

$$\mathcal{E}_2 = \text{Syz}(X^6 - Y^4Z^2, Y^6 - X^2Z^4, X^4Y^2 - Z^6, X^2Y^4, Y^2Z^4, X^4Z^2, X^2Y^2Z^2)(7)$$

on \mathbb{P}^2 from Example 5.6 and Example 5.7. To find a suitable finite morphism $\pi_i : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ as explained above, we try to construct a nontrivial morphism $g_i : \mathbb{P}_{\mathbb{F}_p}^1 \rightarrow \mathbb{P}_{\mathbb{F}_p}^2$ such that the pull-back $g_i^*(\mathcal{E}_i \otimes \mathbb{F}_p)$ is the trivial bundle. Then we can choose a rational map $\pi_i : \mathbb{P}_{\mathbb{Z}_p}^2 \dashrightarrow \mathbb{P}_{\mathbb{Z}_p}^2$ that is defined outside the point $\{[0; 0; 1]\} \in \mathbb{P}_{\mathbb{F}_p}^2 \subset \mathbb{P}_{\mathbb{Z}_p}^2$ such that modulo p there is the commutative diagram

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{F}_p}^2 \setminus \{[0; 0; 1]\} & \xrightarrow{\pi_{i, \mathbb{F}_p}} & \mathbb{P}_{\mathbb{F}_p}^2 \\ \downarrow & \nearrow g_i & \\ \mathbb{P}_{\mathbb{F}_p}^1 & & \end{array}$$

where the vertical morphism is defined as $[X; Y; Z] \mapsto [X; Y]$. It is then clear that $\pi^*(\mathcal{E}_i)$ is modulo p the trivial bundle on the open subset $\mathbb{P}_{\mathbb{F}_p}^2 \setminus \{[0; 0; 1]\}$. A computation with CoCoA shows that the morphism g_i can for example be chosen as $[X; Y] \mapsto [X; Y; 2X + Y]$ if the prime p is odd. Then the rational map π_i can be chosen as $[X; Y; Z] \mapsto [X; Y; 2X + Y + pZ]$. The restriction to the generic fiber gives the finite morphism $\pi_i : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ we were looking for.

On the generic fiber, the bundles $\pi_i^*(\mathcal{E}_i)$ are polystable, and one computes $\dim(\text{End}(\pi_i^*(\mathcal{E}_i))) = h^0(\mathbb{P}^2, \pi_i^*(\mathcal{E}_i) \otimes \pi_i^*(\mathcal{E}_i)) = 1$ using Proposition 4.1. It follows that they have to be stable. Let $\mathfrak{X} \subset \mathbb{P}_{\mathfrak{o}}^2$ be a model of a smooth connected curve $X \subset \mathbb{P}_{\mathbb{Q}_p}^2$ such that the special fiber $\mathfrak{X}_{\mathbb{F}_p}$ does not contain the point $[0; 0; 1]$. If the degree of the plane curve X is large enough, we may use Theorem 6.2 and assume that the restriction of the bundle $\pi_i^*(\mathcal{E}_i)$ is still a stable bundle. It follows from Theorem 7.2 that its Tannaka dual group is a connected semisimple group. Furthermore, we have $\Gamma(X, \pi_i^*(\mathcal{E}_i)^{\otimes 4}|_X) = \Gamma(\mathbb{P}_{\mathbb{Q}_p}^2, f^*(\mathcal{E}_i)^{\otimes 4})$ for curves of sufficiently large degree. Hence the Tannaka dual groups satisfy $G_{\mathcal{E}_1|_X} = \mathbf{Sp}_4 \subset \mathbf{GL}_4$, $G_{\mathcal{E}_2|_X} = \mathbf{Sp}_6 \subset \mathbf{GL}_6$.

Example 7.6. The syzygy bundle

$$\mathcal{E} = \text{Syz}(X^3, Y^3, Z^3, XYZ)(4)$$

on \mathbb{P}^2 is stable of degree 0 due to Theorem 1.1, with Tannaka dual group $G_{\mathcal{E}} = \mathbf{SL}_3 \subset \mathbf{GL}_3$ because of $\dim(\Gamma(\mathbb{P}^2, \mathcal{E}^{\otimes 3})) = 1$. The morphism $g : \mathbb{P}_{\mathbb{F}_p}^2 \rightarrow \mathbb{P}^2$ can be chosen as $[X; Y] \mapsto [X^2 + Y^2; X^2; X^2 + XY]$ and hence $\pi : \mathbb{P}_{\mathbb{Z}_p}^2 \dashrightarrow \mathbb{P}_{\mathbb{Z}_p}^2$ for example as $[X; Y; Z] \mapsto [X^2 + Y^2; X^2; X^2 + XY + pZ^2]$. The same computations and arguments as above then show that $G_{\mathcal{E}|_X} = \mathbf{SL}_3$ for a plane curve X of sufficiently large degree as in the above example.

It is natural to ask if this method works for all semistable vector bundles of degree 0 on \mathbb{P}^N :

Question 7.7. Let \mathcal{E} be a semistable vector bundle of degree 0 on \mathbb{P}^N . Is there always a finite morphism $\pi : \mathbb{P}^N \rightarrow \mathbb{P}^N$ and a model \mathfrak{P} of \mathbb{P}^N such that $\pi^*(\mathcal{E})$ lifts to a sheaf on \mathfrak{P} which is modulo p a free sheaf outside a closed subset of codimension ≥ 2 ?

8. THE STABILITY OF THE SYZYGY BUNDLE OF FIVE GENERIC QUADRICS

It is an open question of whether for generic forms f_1, \dots, f_n of degrees d_1, \dots, d_n in the polynomial ring $R = K[X_0, \dots, X_N]$ over an algebraically closed field K the corresponding syzygy bundle is semistable or even stable. There is no chance if the d_i 's do not satisfy the necessary degree condition of Proposition 3.2. Hence, the question only makes sense if the necessary condition on the degrees is fulfilled, e.g., if we consider forms of constant degree. Since semistability is an open property, it is enough to find a single R_+ -primary family g_1, \dots, g_n having the same degree configuration such that $\text{Syz}(g_1, \dots, g_n)$ is semistable.

Via R_+ -primary monomial families $f_i = X^{\sigma_i}$, $d_i = |\sigma_i|$, one can use Brenner's result 1.1 to establish generic semistability in a combinatorial way by producing examples of monomial families with semistable syzygy bundle. This has been done recently in [36, Theorem 4.6], where P. Macias Marques and R. M. Miró-Roig have proved the stability of the syzygy bundle $\text{Syz}(f_1, \dots, f_n)$ on \mathbb{P}^N for generic forms of degree d with $N+1 \leq n \leq \binom{d+N}{N}$, $(N, d, n) \neq (2, 2, 5)$. This extends [13, Theorem 3.5] of L. Costa, R. M. Miró-Roig and P. Macias Marques, where only the case $N = 2$ has been proven. The general result of [36] has also been obtained simultaneously by I. Coandă in [11]. For the case $N = 2$, $n = 5$ and $d = 2$, where only semistability has been shown, Macias Marques asks the following question (see [35, Problem 2.9]).

Problem 8.1 (Macias Marques). Is there a family of five quadratic homogeneous polynomials in $K[X_0, X_1, X_2]$ such that their syzygy bundle is stable?

Note that one cannot establish generic semistability via a monomial example since for the only candidate we have

$$\mu(\text{Syz}(X^2, Y^2, Z^2, XY, XZ)) = -\frac{5}{2} = \frac{1-6}{2} = \mu(\text{Syz}(X^2, XY, XZ)).$$

We answer Macias Marques's question in the following proposition.

Proposition 8.2. *The syzygy bundle*

$$\text{Syz}(X^2 - Y^2, X^2 - Z^2, XY, XZ, YZ)$$

is stable on $\mathbb{P}^2 = \text{Proj } K[X, Y, Z]$. Moreover, the syzygy bundle for five generic quadrics in $K[X, Y, Z]$ is stable on the projective plane.

Proof. The syzygy bundle $\mathcal{S} = \text{Syz}(X^4 - Y^4, X^4 - Z^4, X^2Y^2, X^2Z^2, Y^2Z^2)$, which we have considered in Example 5.6, is the pull-back of $\mathcal{E} := \text{Syz}(X^2 - Y^2, X^2 - Z^2, XY, XZ, YZ)$ under the finite morphism

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^2, \quad X \longmapsto X^2, \quad Y \longmapsto Y^2, \quad Z \longmapsto Z^2.$$

Since \mathcal{S} is a stable bundle, so is \mathcal{E} . The supplement follows from the openness of stability. \square

Remark 8.3. In the recent preprint [11] I. Coandă has independently proved in [ibid., Example 1.3] the stability of the generic syzygy bundle for $(N, d, n) = (2, 2, 5)$. But his proof is more complicated and does not provide an explicit example of a family of five homogeneous quadrics in three variables.

Let $\mathcal{M}_{\mathbb{P}^N}(n-1, c_1, \dots, c_N)$ be the moduli space of stable vector bundles of rank $n-1$ and Chern classes c_1, \dots, c_N on the projective space \mathbb{P}^N . Denote by $\mathcal{S}_{(N, n, d)} \subset \mathcal{M}_{\mathbb{P}^N}(n-1, c_1, \dots, c_N)$ the stratum of stable syzygy bundles \mathcal{E} defined by the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^N}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0.$$

In his thesis [35], M. Marques computes the dimension of the syzygy stratum and the codimension of its closure in the irreducible component of the moduli space. However, he could not give an answer for the case $N = 2, d = 2, n = 5$ due to the lack of a stable syzygy bundle of five homogeneous quadrics. Our example also closes this gap. We recall that the moduli spaces of stable bundles on \mathbb{P}^2 with fixed invariants are irreducible and their dimensions are known, see for example [17].

Corollary 8.4. *The syzygy stratum $\mathcal{S}_{(2, 5, 2)} \subset \mathcal{M}_{\mathbb{P}^2}(4, -10, 40)$ has dimension 5. In particular, $\overline{\mathcal{S}}_{(2, 5, 2)} = \mathcal{M}_{\mathbb{P}^2}(4, -10, 40)$.*

Proof. See [35, Proposition 4.2 and Theorem 4.3], where the proof presented there works analogously for the case $N = 2, n = 5, d = 2$ due to Proposition 8.2. \square

9. COMPUTING INCLUSION BOUNDS FOR TIGHT CLOSURE AND SOLID CLOSURE

Our semistability algorithm also has impact on certain ideal closure operations in commutative algebra due to a geometric interpretation by H. Brenner. We recall briefly the notions of *tight closure* and *solid closure*, where we restrict ourselves to the case where the ring R under consideration is a Noetherian integral domain. For a detailed exposition of these closure operations and their background see [26].

Let $I = (f_1, \dots, f_n) \subseteq R$ be an ideal and $f \in R$. The R -algebra $A = R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n + f)$ is called the *forcing algebra* for the elements $f_1, \dots, f_n, f \in R$. The element f belongs to the *solid closure*, which we denote by I^* , if and only if the following holds: For every maximal ideal \mathfrak{m} of R the top-dimensional local cohomology module $H_{\mathfrak{m}}^d(A')$ does not vanish, where A' is the forcing algebra for the given data over the local complete domain $R' := \hat{R}_{\mathfrak{m}}$ and $d = \dim(R') = \text{ht}(\mathfrak{m})$.

Now assume that R is of positive characteristic p (i.e., R contains a field of positive characteristic). Then the *tight closure* of I is defined as the ideal

$$I^* := \{f \in R : \text{there exists } 0 \neq t \in R \text{ such that } tf^q \in I^{[q]} \text{ for all } q = p^e\},$$

where $I^{[q]} = (f_1^q, \dots, f_n^q)$ denotes the extended ideal under the e -th iteration of the Frobenius $F : R \rightarrow R, f \mapsto f^p$.

An important fact due to M. Hochster is that $I^* = I^\star$ holds in positive characteristic for a normal K -algebra R of finite type (in fact this is true under weaker assumptions); see [24, Theorem 8.6].

It follows already from the definitions of these closure operations that they are hard to compute. For a normal standard graded integral 2-dimensional algebra R over an algebraically closed field K there is a well-developed theory by H. Brenner for solid closure and tight closure which connects these notions with (strong) semistability of the corresponding syzygy bundle $\text{Syz}(f_1, \dots, f_n)$ on the smooth projective curve $C = \text{Proj } R$; see [8] for an excellent survey. This geometric approach combined with the restriction theorems presented in Section 6 enables us to use our semistability Algorithm 4.6 to compute inclusion bounds for solid closure and tight closure in homogeneous coordinate rings of smooth projective curves, particularly plane curves, of sufficiently large degree.

In characteristic 0 there is the following result for solid closure.

Theorem 9.1 (Brenner). *Let K be an algebraically closed field of characteristic zero and R be a normal standard graded K -domain of dimension two. Further let $I = (f_1, \dots, f_n)$ be an R_+ -primary homogeneous ideal. If $\text{Syz}(f_1, \dots, f_n)$ is semistable on $C = \text{Proj } R$ then*

$$I^* = I + R_{\frac{d_1 + \dots + d_n}{n-1}},$$

where $d_i = \deg(f_i)$ for $i = 1, \dots, n$.

Proof. See the characteristic zero version of [8, Theorem 6.4]. \square

So Theorem 9.1 gives for an element $f \in R_m$ an inclusion $f \in I^*$ for $m \geq \frac{d_1 + \dots + d_n}{n-1}$ and for $m < \frac{d_1 + \dots + d_n}{n-1}$ the question of whether f belongs to I^* reduces to an ideal membership test which is a well-known procedure in computational algebra (cf. for instance [31, Proposition 2.4.10]).

The following theorem works in positive characteristic under the assumption that the syzygy bundle is strongly semistable. We recall that a vector bundle \mathcal{E} is *strongly semistable* if for every $e \geq 0$ the Frobenius pull-backs $F^{e*}(\mathcal{E})$ are semistable. So our algorithmic methods for tight closure can either be applied to homogeneous coordinate rings of the general curve of large degree (via Theorem 6.3) or to coordinate rings of elliptic curves (there semistable bundles are strongly semistable by [38, Theorem 2.1]).

Theorem 9.2 (Brenner). *Let K be an algebraically closed field of characteristic $p > 0$ and R be a normal standard graded K -domain of dimension two. Further let $I = (f_1, \dots, f_n)$ be an R_+ -primary homogeneous ideal such that $\text{Syz}(f_1, \dots, f_n)$ is strongly semistable on $C = \text{Proj } R$. Denote the genus of C by g . Then the following hold.*

- (1) *If $m \geq \frac{d_1 + \dots + d_n}{n-1}$ then $R_m \subseteq I^*$.*
- (2) *If $m < \frac{d_1 + \dots + d_n}{n-1}$ and $f \in R_m$ then $f \in I^*$ if and only if*
 - (a) *$f^p \in I^{[p]} = (f_1^p, \dots, f_n^p)$ if $p > 4(g-1)(n-1)^3$*
 - (b) *or $f^q \in I^{[q]} = (f_1^q, \dots, f_n^q)$ for $q = p^e > 6g$ if $p < 4(g-1)(n-1)^3$.*

Proof. See the positive characteristic version of [8, Theorem 6.4]. \square

Remark 9.3. Let $\mathcal{C} = \text{Proj } R \rightarrow \text{Spec } \mathbb{Z}$ be a generically smooth projective relative curve and $I := (f_1, \dots, f_n)$ be an R_+ -primary ideal. In this situation one can deduce tight closure information of the reductions I_p in the fiber rings $R_p := R \otimes_{\mathbb{Z}} \mathbb{F}_p$ from semistability in characteristic 0. Let $\mathcal{S} := \text{Syz}(f_1, \dots, f_n)$ denote the syzygy bundle on the total space \mathcal{C} . If $\mathcal{S}_0 := \mathcal{S}|_{\mathcal{C}_0}$ is semistable on the generic fiber $\mathcal{C}_0 := \mathcal{C} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Q}$ and $m > \frac{d_1 + \dots + d_n}{n-1}$ then $\mathcal{S}_0(m)$ has positive degree and is therefore ample (see [21, Theorem 2.4]). Since ampleness is an open property, the reductions to positive characteristic $\mathcal{S}_p := \mathcal{S}|_{\mathcal{C}_p}$ and $(\mathcal{S}_p(m))^*$ are also ample on the special fibers $\mathcal{C}_p := \mathcal{C} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{F}_p$ for almost all prime numbers $p \in \mathbb{Z}$. In this situation Brenner's geometric approach also yields results on the tight closure of $I_p \subseteq R_p$ for $p \gg 0$; see [8, Section 4] for a detailed treatment of ampleness criteria for tight closure. In particular, by [ibid., Proposition 4.17] we have $(R_p)_m \subseteq I_p^*$ (in fact $(R_p)_m$ already belongs to the *Frobenius closure* $I_p^F := \{f \in R_p : f^q \in I_p^{[q]} \text{ for some } q = p^e\} \subseteq I_p^*$).

Remark 9.4. What can be said in higher dimensions? As usual, we consider R_+ -primary homogeneous polynomials f_1, \dots, f_n in $P = K[X_0, \dots, X_N]$. We

can compute a minimal graded free resolution \mathfrak{F}_\bullet of the ideal (f_1, \dots, f_n) ; see [32, Section 4.8.B] for the computational background. Since the quotient $R = P/I$ is Artinian, the length of \mathfrak{F}_\bullet equals $N + 1$ by the Auslander-Buchsbaum formula. Consequently, the corresponding resolution of the associated sheaves on \mathbb{P}^N gives a resolution

$$\mathfrak{F}_\bullet : 0 \longrightarrow \mathcal{F}_{N+1} \longrightarrow \mathcal{F}_N \longrightarrow \dots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow 0$$

of the structure sheaf with splitting bundles \mathcal{F}_i , $i = 1, \dots, N + 1$. Instead of looking at $\text{Syz}(f_1, \dots, f_n) = \ker(\mathcal{F}_1 \rightarrow \mathcal{O}_{\mathbb{P}^N})$, we consider the bundle

$$\text{Syz}_{N-1} := \text{Syz}_{N-1}(f_1, \dots, f_n) := \text{im}(\mathcal{F}_{N+1} \longrightarrow \mathcal{F}_N).$$

To check whether Syz_{N-1} is semistable, we can apply Algorithm 4.6 to its dual $(\text{Syz}_{N-1})^*$ which is a kernel bundle. If the answer is positive, then we obtain an inclusion bound for the tight closure $(f_1, \dots, f_n)^*$ in the homogeneous coordinate ring R of a generic hyperplane $X \subset \mathbb{P}^N$ of sufficiently large degree. This works as follows. If we restrict the resolution \mathfrak{F}_\bullet to X , we obtain an exact complex of sheaves on X and the sheaf $\text{Syz}_{N-1}|_X$ is strongly semistable for $k = \deg(X) \gg 0$ by Theorem 6.3. Then Brenner's result [4, Theorem 2.4] gives the inclusion bound $R_{\geq \nu} \subseteq (f_1, \dots, f_n)^*$, where $\nu := -\frac{\mu(\text{Syz}_{N-1}|_X)}{\deg(X)}$. Note that we can compute all necessary invariants (rank, degree, discriminant) of Syz_{N-1} from the resolution \mathfrak{F}_\bullet .

We obtain the same inclusion bound if we restrict Syz_{N-1} to smooth hypersurfaces $X \subset \mathbb{P}^N$ for which every semistable bundle on X is strongly semistable; compare for instance Lemma 2.1 and Example 2.5. Here the degree bound for $\deg(X)$, which ensures the semistability of $\text{Syz}_{N-1}|_X$, is given by Theorem 6.2.

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